Massive perturbations in the Kerr-Newman (anti) de Sitter black hole background

G. V. Kraniotis

University of Ioannina

gkraniot@cc.uoi.gr

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Motivation

Of particular importance are **scalar massive** black hole (BH) perturbations:

- The Klein-Gordon-Fock (KGF) wave equation is the general relativistic version of the Schrödinger equation. It models the interaction of a scalar particle with the gravitational field.
- The discovery of a Higgs-like scalar particle at CERN, proves the existence of scalar particles in Nature.
- The spectacular observation of gravitational waves (GW) from the binary black hole mergers GW150914 and GW151226 by LIGO collaboration, implies that binary black hole systems and their GW emission are abundant in Nature. Elements of physical reality.
- The Kerr-Newman-(anti) de Sitter (KN(a)dS)) metric is the most general exact rotating BH solution of the Einstein-Maxwell equations.

The Kerr-Newman-de Sitter black hole metric

The spacetime interval for the Kerr-Newman-de Sitter black hole solution in Boyer-Lindquist coordinates is ($G = c = 1$):

$$\begin{aligned} \mathrm{d}s^2 = \frac{\Delta_r^{KN}}{\Xi^2\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 - \frac{\rho^2}{\Delta_r^{KN}}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\ - \frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2 \end{aligned} \tag{1}$$

$$\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\theta, \quad \Xi := 1 + \frac{a^2\Lambda}{3}, \tag{2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{3}$$

$$\Delta_r^{KN} := \left(1 - \frac{\Lambda}{3}r^2\right)(r^2+a^2) - 2Mr + e^2, \tag{4}$$

This is accompanied by a non-zero electromagnetic field $F = \mathrm{d}A$ with vector potential :

$$A = -\frac{er}{\Xi(r^2 + a^2\cos^2\theta)}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi). \tag{5}$$

The massive KGF equation in the curved KN(a)dS black hole spacetime

The Klein-Gordon-Fock (KGF) equation for a scalar field Φ that describes the dynamics of a massive scalar electrically charged particle of charge q, in a curved spacetime is described by the equation:

$$\Box\Phi + \mu^2\Phi = 0, \tag{6}$$

where

$$\Box\Phi = \frac{1}{\sqrt{-g}} D_\nu(\sqrt{-g}g^{\mu\nu}D_\mu\Phi). \tag{7}$$

The gauge differential operator is:

$$D_\mu = \partial_\mu - iqA_\mu. \tag{8}$$

By calculating the D'Alembertian of the KGF eqn. in the KNdS BH (initiallly for $q = 0$):

The massive KGF equation in the curved KN(a)dS black hole spacetime

$$\frac{\Xi^2}{\rho^2}\left[\frac{(r^2+a^2)^2}{\Delta_r^{KN}}-\frac{a^2\sin^2\theta}{\Delta_\theta}\right]\frac{\partial^2\Phi}{\partial t^2}-\frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\Delta_r^{KN}\frac{\partial\Phi}{\partial r}\right)-\frac{1}{\rho^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\Delta_\theta\frac{\partial\Phi}{\partial\theta}\right)$$
$$+2\frac{a\Xi^2}{\rho^2}\left\{-\frac{1}{\Delta_\theta}+\frac{r^2+a^2}{\Delta_r^{KN}}\right\}\frac{\partial^2\Phi}{\partial t\partial\phi}-\frac{\Xi^2}{\rho^2\sin^2\theta}\left\{\frac{1}{\Delta_\theta}-\frac{a^2\sin^2\theta}{\Delta_r^{KN}}\right\}\frac{\partial^2\Phi}{\partial\phi^2}+\mu^2\Phi=0. \quad (9)$$

To solve (9) we assume the ansatz:

$$\Phi=\Phi(\vec{r},t)=R(r)S(\theta)e^{im\varphi}e^{-i\omega t}. \quad (10)$$

Substituting (10) into (9) we obtain:

$$\frac{1}{R(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta_r^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\Xi^2\left[\frac{(r^2+a^2)^2}{\Delta_r^{KN}}-\frac{a^2\sin^2\theta}{\Delta_\theta}\right](-\omega^2)$$
$$+\frac{1}{S(\theta)}\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\Delta_\theta\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\right)+\frac{\Xi^2}{\sin^2\theta}\left\{\frac{1}{\Delta_\theta}-\frac{a^2\sin^2\theta}{\Delta_r^{KN}}\right\}(-m^2)$$
$$-2a\Xi^2\left\{-\frac{1}{\Delta_\theta}+\frac{r^2+a^2}{\Delta_r^{KN}}\right\}m\omega-\rho^2\mu^2=0, \quad (11)$$

The massive KGF equation in the curved KN(a)dS black hole spacetime separates

In our CQG paper we proved the *separation* of radial from angular parts that yields the DEs (G. V. Kraniotis 2016, Class.Quantum Grav.33 225011):

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\Delta_\theta\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\right) + S(\theta)\left[-\frac{m^2\Xi^2}{\sin^2\theta}\frac{1}{\Delta_\theta}+\frac{2a\Xi^2}{\Delta_\theta}m\omega-\frac{\Xi^2 a^2\sin^2\theta\omega^2}{\Delta_\theta}-\mu^2a^2\cos^2\theta+K_{lm}\right]=0, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta_r^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{R(r)}{\Delta_r^{KN}}[\Xi^2K^2-r^2\mu^2\Delta_r^{KN}-K_{lm}\Delta_r^{KN}]=0, \tag{13}$$

where K_{lm} denotes the separability constant and

$$K(r):=\omega(r^2+a^2)-am. \tag{14}$$

Now including the contribution from the electric charge of the scalar particle we calculate the *modified* radial Fuchsian differential equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta_r^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{R(r)}{\Delta_r^{KN}}\left[\Xi^2\left(K-\frac{eqr}{\Xi}\right)^2-r^2\mu^2\Delta_r^{KN}-K_{lm}\Delta_r^{KN}\right]=0} \tag{15}$$

while the angular equation remains unaltered.

Heun's differential equation

Introduced by K. Heun as a generalisation of the hypergeometric equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0. \tag{16}$$

In (16), y and z are regarded as complex variables and the parameters $\alpha, \beta, \gamma, \delta, \varepsilon, q, a \in \mathbb{C}$ except that $a \in \mathbb{C} \setminus \{0, 1\}$. The first five parameters are linked by the equation $\gamma + \delta + \varepsilon = \alpha + \beta + 1$ Heun's equation (HE) is thus of Fuchsian type with 4 regular singularities at the points $z = 0, 1, a, \infty$, with *exponents*: $\{0, 1-\gamma\}; \{0, 1-\delta\}; \{0, 1-\varepsilon\}; \{\alpha, \beta\}$. HE includes an *accessory* or *auxiliary* parameter, q, which in many applications appears as a spectral parameter. By merging the singularity at $z = a$ of HE with that at $z = \infty$, yields the *confluent Heun equation* (CHE)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \nu\right]\frac{\mathrm{d}y}{\mathrm{d}z} + \left[\frac{\alpha\nu z - \sigma}{z(z-1)}\right] y(z) = 0. \tag{17}$$

Eqn (17) still has regular singularities at $z = 0$ and $z = 1$, and an *irregular* singularity of rank 1 at $z = \infty$ (RONVEAUX). In (17) γ, δ, α are the same parameters as in HE (16) while ν, σ are new. CHE

Open issues on the Heun project

- Despite the phenomenal simplicity of HE the theory of its solutions (the so called Heun functions) is far from being complete.
- A lot of research effort has been invested on the so called (still open) connection problem, i.e. the problem of finding relations between local solutions of HE about two different singularities.
- A series of open issues require further insight, among others we mention, the details of the monodromy group and finding integral representations for the Heun solutions.

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

The angular differential equation was determined to be:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\Delta_\theta\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\right) + S(\theta)\left[\frac{-m^2\Xi^2}{(\sin\theta)^2}\frac{1}{\Delta_\theta} - \frac{\Xi^2a^2\sin^2\theta\omega^2}{\Delta_\theta} + \frac{2a\Xi^2m\omega}{\Delta_\theta} - \mu^2a^2\cos^2\theta\right] = -K_{lm}S(\theta) \tag{18}$$

By defining the variable $x := \cos\theta$, and setting $\mu = \sqrt{\frac{2\Lambda}{3}}, \Lambda > 0$, equation (18) becomes:

$$\left[\left(1+\frac{a^2\Lambda}{3}x^2\right)(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{a^2\Lambda}{3}x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} - 2\left(1+\frac{a^2\Lambda}{3}x^2\right)x\frac{\mathrm{d}}{\mathrm{d}x}\right]S + \left[-\frac{\Xi^2a^2\omega^2(1-x^2)}{1+\frac{a^2\Lambda}{3}x^2} + \frac{2a\omega m\Xi^2}{1+\frac{a^2\Lambda}{3}x^2} - \frac{m^2\Xi^2}{(1+\frac{a^2\Lambda}{3}x^2)(1-x^2)}\right]S + \left[-2\frac{a^2\Lambda}{3}x^2 + K_{lm}\right]S = 0 \tag{19}$$

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

The angular Fuchsian equation (19) has four regular singularities at the points $\pm 1, \pm \frac{i}{\sqrt{\alpha_\Lambda}}$, which we denote with the tuple $(a_1, a_2, a_3, a_4) = (-1, 1, -\frac{i}{\sqrt{\alpha_\Lambda}}, \frac{i}{\sqrt{\alpha_\Lambda}})$. The automorphism group of the parameter space of HE has been determined, thus we apply first to equation (19) the *homographic transformation of the independent variable* :

$$\boxed{z = \frac{a_2 - a_4}{a_2 - a_1}\frac{x - a_1}{x - a_4} = \frac{1 - \frac{i}{\sqrt{\alpha_\lambda}}}{2}\frac{x + 1}{x - \frac{i}{\sqrt{\alpha_\Lambda}}}, \quad \alpha_\Lambda := \frac{a^2\Lambda}{3},} \tag{20}$$

where such a transformation is designed to map the three singularities a_1, a_2, a_4 into $0, 1, \infty$. The fourth singularity $a_3 \overset{(20)}{\rightarrow} z_3 = \frac{a_3 - a_1}{a_3 - a_4}\frac{a_2 - a_4}{a_2 - a_1}$. With this transformation we have:

$$(1 + \alpha_\Lambda x^2)(1 - x^2) = \frac{\alpha_\Lambda 16 i \Xi^2}{\sqrt{\alpha_\Lambda}} \frac{z(z-1)(z-z_3)}{[2z\sqrt{\alpha_\Lambda} - \sqrt{\alpha_\Lambda} + i]^4}, \tag{21}$$

where

$$z_3 = -\frac{1}{2}\left(-1 + \frac{\alpha_\Lambda - 1}{2i\sqrt{\alpha_\Lambda}}\right). \tag{22}$$

Equation (19) with the aid of (20) becomes:

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

$$\begin{aligned}&\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}\\
&-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{2}{(z-z_\infty)^2}\\
&+\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]\\
&+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}+\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}+\frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]\\
&+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]\\
&+\frac{1}{z-z_\infty}\frac{-8i\sqrt{\alpha_\Lambda}}{\Xi}\Bigg\}S(z)=0, \qquad (23)\end{aligned}$$

where $z_\infty=-\frac{-i(1+\sqrt{\alpha_\Lambda}i)}{2\sqrt{\alpha_\Lambda}}$ and $\xi:=a\omega$. The four singularities $z=0,1,z_3,z_\infty$ have exponents:
$\{\frac{|m|}{2},-\frac{|m|}{2}\},\{\frac{|m|}{2},-\frac{|m|}{2}\},\{\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}}-m\sqrt{\alpha_\Lambda}\right),-\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}}-m\sqrt{\alpha_\Lambda}\right)\},\{2,1\}$.
Thus equation (23) is *not* of a Heun type.

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

The *F-homotopic transformation* or *index transformation* of the dependent variable S:

$$S(z) = z^{\alpha_1}(z-1)^{\alpha_2}(z-z_3)^{\alpha_3}(z-z_\infty)^{\alpha_4}\bar{S}(z) \tag{24}$$

where $\alpha_1 = \alpha_2 = \frac{|m|}{2}, \alpha_3 = \pm\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}} - m\sqrt{\alpha_\Lambda}\right), \alpha_4 = 1$ is designed to reduce one of the exponents of the finite singularities $0, 1, z_3$ to zero and to eliminate the finite z_∞ singularity. In other words transforms (23) into the Heun form (16). Indeed application of (24) into (23) yields:

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{2\alpha_1+1}{z} + \frac{2\alpha_2+1}{z-1} + \frac{2\alpha_3+1}{z-z_3}\right]\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-z_3)}\right\}\bar{S}(z) = 0, \tag{25}$$

where the *auxiliary parameter q* is calculated in terms of the cosmological constant, spin of the black hole,the parameters m, ω and is given by the expression:

$$q = \frac{i}{4\sqrt{\alpha_\Lambda}}\Big\{-(1+i\sqrt{\alpha_\Lambda})^2[2\alpha_1\alpha_2+\alpha_2+\alpha_1] - 4\sqrt{\alpha_\Lambda}i[2\alpha_1\alpha_3+\alpha_3+\alpha_1]$$
$$-\frac{m^2}{2}((1+i\sqrt{\alpha_\Lambda})^2+4\alpha_\Lambda) + K_{lm} - 2i\sqrt{\alpha_\Lambda} + 2\Xi m\xi\Big\}$$

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

The parameters α, β are given in terms of the physical parameters by the expression:

$$
\begin{aligned}
\alpha\beta &= q - (z_3 - 1) \times \mathcal{B} \\
&= \frac{i}{4\sqrt{\alpha_\Lambda}} \Big\{ -(1 + i\sqrt{\alpha_\Lambda})^2 [2\alpha_1\alpha_2 + \alpha_2 + \alpha_1] - 4\sqrt{\alpha_\Lambda} i [2\alpha_1\alpha_3 + \alpha_3 + \alpha_1] \\
&- \frac{m^2}{2} ((1 + i\sqrt{\alpha_\Lambda})^2 + 4\alpha_\Lambda) + K_{lm} - 2i\sqrt{\alpha_\Lambda} + 2\Xi m\xi \Big\} \\
&+ \frac{i}{4\sqrt{\alpha_\Lambda}} \Big\{ \frac{m^2}{2} ((1 - i\sqrt{\alpha_\Lambda})^2 + 4\alpha_\Lambda) - 2m\xi\Xi - K_{lm} - 2\sqrt{\alpha_\Lambda} i \\
&+ (1 - i\sqrt{\alpha_\Lambda})^2 [2\alpha_1\alpha_2 + \alpha_2 + \alpha_1] + i4\sqrt{\alpha_\Lambda}(-2\alpha_2\alpha_3 - \alpha_3 - \alpha_2) \Big\}
\end{aligned}
\tag{27}
$$

The parameter $\mathcal{B}$ is the total coefficient of the term $\frac{1}{z-1}$ that results after the application of the F-homotopic transformation (24) in (23).

Closed form solution of the radial equation for a massive charged particle in the KNdS black hole spacetime in terms of Heun functions for specific values of the scalar mass

The massive radial Fuchsian equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta_r^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{R(r)}{\Delta_r^{KN}}\left[\Xi^2\left(K-\frac{eqr}{\Xi}\right)^2-r^2\mu^2\Delta_r^{KN}-K_{lm}\Delta_r^{KN}\right]=0} \tag{28}$$

We write the quantity Δ_r^{KN} in terms of the radii of the event and Cauchy horizons r_+, r_- and the cosmological horizon r_Λ^+ for positive Λ:

$$\Delta_r^{KN}=-\frac{\Lambda}{3}(r-r_+)(r-r_-)(r-r_\Lambda^+)(r-r_\Lambda^-) \tag{29}$$

There are five regular singularities in (28), at the points $r_\pm, r_\Lambda^\pm, \infty$. Applying the homographic substitution

$$\boxed{z=\left(\frac{r_+-r_\Lambda^-}{r_+-r_-}\right)\left(\frac{r-r_-}{r-r_\Lambda^-}\right)} \tag{30}$$

Exact solution for the massive-charged radial KGF Fuchsian equation

Equation (29) in terms of the new variable is written:

$$\boxed{\Delta_r^{KN} = -\frac{\Lambda}{3}\frac{H z_\infty^3 z(z-1)(z-z_r)}{(z_\infty - z)^4},} \tag{31}$$

where $H := \frac{(r_- - r_\Lambda^-)^2 (r_+ - r_-)(r_\Lambda^+ - r_-)}{z_r}$. Also we have the following relations:

$$r = \frac{r_- z_\infty - r_\Lambda^- z}{z_\infty - z}, \tag{32}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{z_\infty (r_- - r_\Lambda^-)}{(r - r_\Lambda^-)^2} = \frac{1}{z_\infty}\frac{1}{r_- - r_\Lambda^-}(z_\infty - z)^2 = \frac{r_+ - r_-}{r_+ - r_\Lambda^-}\frac{1}{r_- - r_\Lambda^-}(z_\infty - z)^2 \tag{33}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = \frac{-2 z_\infty (r_- - r_\Lambda^-)}{(r - r_\Lambda^-)^3}, \quad \frac{\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2} = \frac{-2}{z_\infty - z}. \tag{34}$$

The quantities z_∞, z_r are defined as follows:

$$z_\infty := \frac{r_+ - r_\Lambda^-}{r_+ - r_-}, \quad z_r := z_\infty \left(\frac{r_\Lambda^+ - r_-}{r_\Lambda^+ - r_\Lambda^-}\right). \tag{35}$$

Applying the homographic transformation (30) in the radial equation for a massive charged particle (15) we obtain:

$$
\begin{aligned}
&\frac{\mathrm{d}^2 R}{\mathrm{d}z^2}+\frac{1}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}\frac{1}{\Delta_r^{KN}}\frac{\mathrm{d}\Delta_r^{KN}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r}+\frac{\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)}\frac{\mathrm{d}R}{\mathrm{d}z}\\
&+\frac{\Xi^2(K(r)-\frac{eqr}{\Xi})^2}{(\Delta_r^{KN})^2\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}R+\frac{-r^2\mu^2 R}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Delta_r^{KN}}-\frac{K_{lm}R}{\Delta_r^{KN}\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}\\
=&\frac{\mathrm{d}^2 R}{\mathrm{d}z^2}+\left\{\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_r}-\frac{2}{z-z_\infty}\right\}\frac{\mathrm{d}R}{\mathrm{d}z}\\
&+\left[\frac{A'}{z^2}+\frac{B'}{z}+\frac{C'}{(z-1)^2}+\frac{D'}{z-1}+\frac{E'}{(z-z_r)^2}+\frac{H'}{z-z_r}\right]R\\
&+\left[\frac{A}{(z_\infty-z)^2}+\frac{B}{z_\infty-z}+\frac{C}{z}+\frac{D}{z-1}+\frac{F}{z-z_r}\right]R\\
&+\left[\frac{\mathcal{B}_{K_{lm}}}{z}+\frac{\mathcal{D}_{K_{lm}}}{z-1}++\frac{\mathcal{H}_{K_{lm}}}{z-z_r}\right]R=0. \qquad (36)
\end{aligned}
$$

Exact solution for the massive-charged radial KGF Fuchsian equation

where we compute the coefficients of the expansion as follows:

$$A = \frac{3\mu^2}{\Lambda}, \tag{37}$$

$$B = \frac{3\mu^2}{\Lambda}\frac{1}{r_- - r_\Lambda^-}\left[\frac{(r_\Lambda^- + r_-)z_r - 2r_- z_\infty - 2r_- z_r z_\infty - (r_\Lambda^- - 3r_-)z_\infty^2}{(1-z_\infty)(z_r - z_\infty)z_\infty}\right], \tag{38}$$

$$C = \frac{3\mu^2}{\Lambda}\frac{1}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{r_-^2}{z_\infty}, \tag{39}$$

$$D = -\frac{3\mu^2}{\Lambda}\frac{z_r}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{1}{z_\infty}\frac{[r_\Lambda^- - r_- z_\infty]^2}{(z_r - 1)(z_\infty - 1)}, \tag{40}$$

$$F = \frac{3\mu^2}{\Lambda}\frac{1}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{1}{z_\infty}\frac{(r_\Lambda^- z_r - r_- z_\infty)^2}{(z_r - 1)(z_r - z_\infty)^2} \tag{41}$$

while the expansion coefficients A', C', E' are computed to be:

$$A' = \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_-) - eqr_-]^2}{(r_- - r_\Lambda^-)^2(r_+ - r_-)^2(r_\Lambda^+ - r_-)^2} \tag{42}$$

$$C' = \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_+) - eqr_+]^2}{(r_+ - r_\Lambda^-)^2(r_+ - r_\Lambda^+)^2(r_+ - r_-)^2} \tag{43}$$

$$E' = \frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_\Lambda^+) - eqr_\Lambda^+]^2}{(r_\Lambda^+ - r_-)^2(r_\Lambda^+ - r_\Lambda^-)^2(r_+ - r_\Lambda^+)^2} \tag{44}$$

Let us calculate the exponents of the singularity at z_∞. The indicial equation takes the form:

$$F(r) = r(r-1) - 2r + \frac{3\mu^2}{\Lambda} = 0, \tag{45}$$

and the exponents are computed to be:

$$r_{\mu z_\infty}^{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - \frac{12\mu^2}{\Lambda}}. \tag{46}$$

Calculation of exponents for the F-homotopic radial transformation

Subsequently we compute the exponents for the regular singularities $z=0, z=1, z=z_r$. Indeed the indicial equation for the $z=1$ singularity takes the form:

$$F(r)=r(r-1)+r+\frac{a^4}{\alpha_\Lambda^2}\frac{[\Xi K(r_+)-eqr_+]^2}{(r_+-r_\Lambda^-)^2(r_+-r_\Lambda^+)^2(r_+-r_-)^2}=0 \qquad (47)$$

Thus the roots are calculated to be:

$$\boxed{r_{z=1}^{1,2}\equiv\mu_2=\pm\frac{ia^2}{\alpha_\Lambda}\frac{\Xi K(r_+)-eqr_+}{(r_\Lambda^- -r_+)(r_- -r_+)(r_\Lambda^+ -r_+)}} \qquad (48)$$

Likewise we compute the exponents of the other two singularities:

$$r_{z=0}^{1,2}\equiv\mu_1=\pm\frac{ia^2}{\alpha_\Lambda}\frac{\Xi K(r_-)-eqr_-}{(r_- -r_\Lambda^-)(r_+ -r_-)(r_\Lambda^+ -r_-)}, \qquad (49a)$$

$$r_{z=z_r}^{1,2}\equiv\mu_3=\pm\frac{ia^2}{\alpha_\Lambda}\frac{[\Xi K(r_\Lambda^+)-eqr_\Lambda^+]}{(r_\Lambda^- -r_\Lambda^+)(r_+ -r_\Lambda^+)(r_- -r_\Lambda^+)}. \qquad (49b)$$

Thus we see that in general the massive radial Fuchsian KGF equation for a charged particle in the curved spacetime of a cosmological rotating charged black hole possess five singularities including the infinity.

Choosing a value of the scalar mass in terms of Λ as $\mu=\sqrt{\frac{2}{3}\Lambda}$ the exponents of the z_∞ singularity become $r^{1,2}_{z_\infty,\mu^2=\frac{2}{3}\Lambda}=2,1$. Thus applying the F-homotopic transformation of the dependent variable R

$$\boxed{R(z)=z^{\mu_1}(z-1)^{\mu_2}(z-z_r)^{\mu_3}(z-z_\infty)^{r^2_{z_\infty}}\bar{R}(z)} \tag{50}$$

we *eliminate the z_∞ singularity* and reduce one of the exponents of the three finite singularities $z=0,1,z_r$ to zero. Consequently for this value for the scalar mass the radial part of the KGF Fuchsian equation in the curved spacetime of the KNdS black hole becomes a Heun differential equation:

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{2\mu_1+1}{z}+\frac{2\mu_2+1}{z-1}+\frac{2\mu_3+1}{z-z_r}\right]\frac{\mathrm{d}}{\mathrm{d}z}\right.\\&\left.+\frac{\alpha\beta z-q}{z(z-1)(z-z_r)}\right\}\bar{R}(z)=0.\end{aligned} \tag{51}$$

The $F-$ *homotopic transformation* (50) *factors out* the z_∞ singularity because it eliminates both terms $\propto \frac{1}{(z-z_\infty)^2}$ and $\propto \frac{1}{z-z_\infty}$ respectively. Indeed the last term vanishes:

$$\begin{aligned}&\frac{1}{z-z_\infty}\left(\frac{1}{z_\infty}-\frac{1}{1-z_\infty}-\frac{1}{z_r-z_\infty}\right)-\frac{B}{z-z_\infty}\\&=\frac{1}{z-z_\infty}\frac{(r_--r_+)(r_\Lambda^-+r_\Lambda^++r_-+r_+)}{(r_\Lambda^--r_-)(r_\Lambda^--r_+)}=0,\end{aligned} \tag{52}$$

due to Vieta's relations, i.e. $r_\Lambda^-+r_\Lambda^++r_-+r_+=0$.

Theorem

For the value of the scalar mass parameter: $\mu = \sqrt{\frac{2\Lambda}{3}}$ *both radial and angular Fuchsian differential equations that result from separation of variables of the KGF equation in KNdS spacetime, are transformed into Heun's equations by eliminating the singularity at* z_∞.

Therefore both equations can be solved in closed analytic form in terms of *general Heun functions*. Radial $\bar{R}(z)$ and angular parts $\bar{S}(z)$ can be expressed locally in terms of Heun functions:
$HI(a_i, q_i; \alpha_i, \beta_i, \gamma_i, \delta_i; z), \quad i = \bar{R}, \bar{S}$.

Exact solution of the Heun angular equation in KNdS spacetime in hypergeometric polynomials

For $u(z)$ a function that satisfies Heun's equation, we make the following ansatz:

$$u(z) = \sum_{\nu=0}^{\infty} c_\nu y_\nu(z) \tag{53}$$

where

$$y_\nu(z) = F(-\nu, \nu+\omega, \gamma, z) = \frac{\nu!\Gamma(\gamma)}{\Gamma(\nu+\gamma)} P_\nu^{(\gamma-1,\omega-\gamma)}(1-2z) \tag{54}$$

In eqn.(54), $\omega = \delta + \gamma - 1$ and should not be confused with the angular frequency that appears in the separation ansatz. The polynomials satisfy the differential equation

$$y_\nu''(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1}\right] y_\nu'(z) - \frac{\nu(\nu+\omega)}{z(z-1)} y_\nu(z) = 0 \tag{55}$$

and the recursion relations $\nu \in \mathbb{Z}^+$:

$$zy_\nu(z) = P_\nu y_{\nu+1}(z) + Q_\nu y_\nu(z) + R_\nu y_{\nu-1}(z), \tag{56}$$

$$z(z-1)\frac{\mathrm{d}}{\mathrm{d}z} y_\nu(z) = P'_\nu y_{\nu+1}(z) + Q'_\nu(z) y_\nu(z) + R'_\nu y_{\nu-1}(z) \tag{57}$$

where $(\nu \in \mathbb{Z}^+)$

$$\left\{ \begin{array}{l} P_\nu = -\frac{(\nu+\omega)(\nu+\gamma)}{(2\nu+\omega)(2\nu+\omega+1)} \\ Q_\nu = \frac{(\omega-1)(\gamma-\delta)}{2(2\nu+\omega+1)(2\nu+\omega-1)} + \frac{1}{2} \\ R_\nu = -\frac{\nu(\nu+\delta-1)}{(2\nu+\omega)(2\nu+\omega-1)} \end{array} \right. \quad \left\{ \begin{array}{l} P'_\nu = -\frac{\nu(\nu+\omega)(\nu+\gamma)}{(2\nu+\omega)(2\nu+\omega+1)} \\ Q'_\nu = \frac{\nu(\nu+\omega)(\gamma-\delta)}{(2\nu+\omega+1)(2\nu+\omega-1)} \\ R'_\nu = \frac{\nu(\nu+\omega)(\nu+\delta-1)}{(2\nu+\omega)(2\nu+\omega-1)} \end{array} \right.$$

with initialising values $(\nu = 0)$

$$\left\{ \begin{array}{l} P_0 = -\frac{\gamma}{\omega+1} \\ Q_0 = \frac{\gamma}{\omega+1} \\ R_0 = 0 \end{array} \right. \quad \left\{ \begin{array}{l} P'_0 = 0 \\ Q'_0 = 0 \\ R'_0 = 0 \end{array} \right.$$

Inserting (53) into Heun's equation (16) we find that if the series (53) is a solution of the Heun's differential equation the coefficients in the series of hypergeometric polynomials, c_ν, satisfy:

$$c_{\nu+1} = E_\nu c_\nu + F_\nu c_{\nu-1}, \tag{58}$$

where

$$\begin{cases} E_\nu = -\frac{[\nu(\nu+\omega)+\alpha\beta]Q_\nu+\varepsilon Q'_\nu - a\nu(\nu+\omega)-q}{\varepsilon R'_{\nu+1}+[(\nu+1)(\nu+1+\omega)+\alpha\beta]R_{\nu+1}} \\ F_\nu = -\frac{\varepsilon P'_{\nu-1}+((\nu-1)(\nu-1+\omega)+\alpha\beta)P_{\nu-1}}{\varepsilon R'_{\nu+1}+[(\nu+1)(\nu+1+\omega)+\alpha\beta]R_{\nu+1}} \end{cases}$$

The sequence is initialised by

$$c_1 = -\frac{c_0(\alpha\beta\gamma - q(\omega+1))(2+\omega)}{\delta((1+\omega)(\varepsilon-1)-\alpha\beta)} \tag{59}$$

Convergence

Using the first of the recursion relations and the well defined limits of the coefficients P_ν, Q_ν, R_ν as $\nu \to \infty$:

$$\begin{cases} P_\nu = -\frac{1}{4} + \frac{1-2\gamma}{8\nu} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \\ Q_\nu = \frac{1}{2} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \quad \text{as } \nu \to \infty \\ R_\nu = -\frac{1}{4} - \frac{1-2\gamma}{8\nu} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \end{cases}$$

one can show that

Lemma

$$\lim_{\nu\to\infty} \frac{y_{\nu+1}(z)}{y_\nu(z)} = s(z) \equiv s_2(z) = \frac{(1-z^{-1})^{1/2}+1}{(1-z^{-1})^{1/2}-1} \equiv \frac{Z+1}{Z-1} \tag{60}$$

exists where the branch of the square root satisfies $\Re(1-z^{-1})^{1/2} > 0$ *(assuming* $z \notin [0,1]$*). Moreover,*
$\frac{y_{\nu+1}(z)}{y_\nu(z)} = s(z) + \frac{\sigma(z)}{\nu} + \mathcal{O}(1/\nu^2)$, as $\nu \to \infty$, $\sigma(z) = s(z)\frac{1-2\gamma}{2}$.

(the exceptional case $\lim_{\nu\to\infty} \frac{y_{\nu+1}(z)}{y_\nu(z)} = s_1(z) = \frac{1}{s(z)}$ does not occur Erdélyi QJM 1944). The asymptotic behaviour of the coefficients c_ν is given as follows:

Lemma

The limit

$$\lim_{\nu\to\infty}\frac{c_{\nu+1}}{c_\nu}=t_2=\frac{(1-a^{-1})^{1/2}+1}{(1-a^{-1})^{1/2}-1}\equiv\frac{A+1}{A-1},\tag{61}$$

exists, where the branch of the square root satisfies $\Re A=\Re(1-a^{-1})^{1/2}>0$ *(under the assumption* $|a|>1$*), and* $|t_2|>1$*. In the exceptional case, the limit is*

$$\lim_{\nu\to\infty}\frac{c_{\nu+1}}{c_\nu}=t_1=\frac{(1-a^{-1})^{1/2}-1}{(1-a^{-1})^{1/2}+1}\equiv\frac{A-1}{A+1}\tag{62}$$

and $|t_1|<1$*. Moreover, if* $\frac{c_{\nu+1}}{c_\nu}=t_n+\frac{\tau}{\nu}+\mathcal{O}(1/\nu^2)$, as $\nu\to\infty$ $(n=1,2)$ *then* τ *satisfies:* $\tau=t_n(2\gamma+2\varepsilon-5)\frac{(1-2a)t_n-1}{t_n^2-1}$, $n=1,2$.

By the D'Alembert's ratio test absolute convergence of the series (53) is guaranteed provided

$$\lim_{\nu\to\infty}\left|\frac{c_{\nu+1}y_{\nu+1}(z)}{c_\nu y_\nu(z)}\right|=|t_n s_2(z)|<1,\ \ n=1,2\tag{63}$$

and diverges if $|t_n s_2(z)|>1,\ n=1,2$.

In general t_2 is the proper root for the asymptotic limit of the coefficients. The boundary of the domain of convergence, $|t_2 s_2(z)| = 1$, consists of all $z \in \mathbb{C}$ satisfying:

$$\left|\frac{Z+1}{Z-1}\right| = |s_2(z)| = \frac{1}{|t_2|} = \left|\frac{A-1}{A+1}\right| < 1 \tag{64}$$

The solution set is void since $|Z+1| < |Z-1| \equiv \Re Z < 0$ which contradicts the assumption $\Re Z > 0$. The other root, t_1, gives a domain of convergence determined [1] by

$$\left|\frac{Z+1}{Z-1}\right| = |s_2(z)| < \frac{1}{|t_1|} = \left|\frac{A+1}{A-1}\right| \tag{65}$$

which defines the interior of an ellipse in the complex z-plane, with foci at $z = 0, 1$ and passing through $z = a$. On the ellipse $t_1 s_2(z) = 1$. Raabe's test guarantees absolute convergence, if there exists $c > 0$ such that:

$$\lim_{\nu\to\infty} \nu \Re\left(\frac{c_{\nu+1} y_{\nu+1}(z)}{c_\nu y_\nu(z)} - 1\right) = -1 - c \tag{66}$$

[1] The exceptional limit to the root t_1 is also known in the literature as the phenomenon of augmented convergence.

From the results above

$$\lim_{\nu\to\infty} \nu\Re\left(\frac{c_{\nu+1}y_{\nu+1}(z)}{c_\nu y_\nu(z)}-1\right)=\lim_{\nu\to\infty}\nu\Re\left((t_1+\frac{\tau}{\nu})(s_2(z)+\frac{\sigma(z)}{\nu})-1\right)$$
$$=\Re(t_1\sigma(z)+\tau s_2(z))$$
$$=\Re(t_1 s_2(z)\frac{1-2\gamma}{2}+t_1(2\gamma+2\varepsilon-5)\frac{(1-2a)t_1-1}{t_1^2-1}s_2)$$
$$=\Re\varepsilon-2. \tag{67}$$

Thus by Raabe's test the series converges absolutely on the ellipse if $\Re\varepsilon<1$. The formal procedure can be applied to angular Heun equation with the ansatz: $\bar{S}(z)=\sum_{\nu=0}^{\infty}c_\nu y_\nu(z)$ where $y_\nu(z)$ are given in (54) and $a=z_3$. The parameters of angular Heun's equation are given in (25),(26), $\alpha=\sum_{i=1}^{3}\alpha_i+\alpha_3^*+1$, $\beta=\sum_{i=1}^{3}\alpha_i-\alpha_3^*+1$. The separability constant K_{lm} can be determined from the recurrence relation (58) compatible with the augmented convergence of the series expansion (53) solution (G. V. Kraniotis 2016, Class.Quantum Grav.33 225011).

Solution of the massive radial equation in KNdS spacetime in the ellipse with foci at the event and Cauchy horizons

Thus we proved using the concept of augmented convergence that the angular equation in KNdS BH can be solved in terms of an infinite series of Jacobi polynomials which converges inside the ellipse with foci at $z=0$ and $z=1$ passing through the point z_3 with possible exception of the line connecting the two foci. We also mentioned how the separability constant K_{lm} can in principle be determined in a compatible way with augmented convergence. For the massive radial equation for a charged scalar in KNdS spacetime with $\mu=\sqrt{\frac{2\Lambda}{3}}$, the analytic solution convergent in the ellipse with foci at $z=0,1$ which correspond to $r=r_-,r_+$, respectively, is given by

$$\bar{R}_\nu(z)=\sum_{\varrho=-\infty}^{+\infty} c_\varrho^\nu u_{\nu+\varrho}(z), \tag{68}$$

$$u_\nu(z)=F(-\nu,\nu+2(\mu_1+\mu_2)+1,2\mu_1+1,z) \tag{69}$$

The expansion coefficients c_ϱ^ν are determined by the recurrence relation (58) written as follows:

$$D_\varrho^\nu c_{\varrho+1}^\nu + E_\varrho^\nu c_\varrho^\nu + F_\varrho^\nu c_{\varrho-1}^\nu = 0, \tag{70}$$

where

$$D_\varrho^\nu = -\frac{(\nu+\varrho+\delta)(\nu+\varrho+1)(\nu+\varrho+1+\omega-\alpha)(\nu+\varrho+1+\omega-\beta)}{(2\nu+2\varrho+\omega+2)(2\nu+2\varrho+\omega+1)}, \tag{71}$$

$$F_\varrho^\nu = -\frac{(\nu+\varrho-1+\omega)(\nu+\varrho-1+\gamma)(\nu+\varrho-1+\alpha)(\nu+\varrho-1+\beta)}{(2\nu+2\varrho+\omega-2)(2\nu+2\varrho+\omega-1)}, \tag{72}$$

$$E_\varrho^\nu = \frac{J_\varrho^\nu}{(2\nu+2\varrho+\omega+1)(2\nu+2\varrho+\omega-1)} - z_r(\nu+\varrho)(\nu+\varrho+\omega) - q, \tag{73}$$

$$\begin{aligned} J_\varrho^\nu = {} & [(\nu+\varrho)(\nu+\varrho+\omega)+\alpha\beta](2(\nu+\varrho)(\nu+\varrho+\omega)+\gamma(\omega-1)) \\ & + \varepsilon(\nu+\varrho)(\nu+\varrho+\omega)(\gamma-\delta) \end{aligned} \tag{74}$$

The radial solution is expressed as a series, where ϱ runs from $-\infty$ to ∞, because $\nu \notin \mathbb{Z}$, since the separation constant K_{lm} is fixed from the angular solution and the parameter ν is determined by the corresponding in the radial case transcendental equation.

As if this was not enough, physics of black holes points to a generalisation of Heun functions

- The general case, however, is that the solution of the KGF equation with the method of separation of variables, for a rotating charged cosmological black hole, results in FDE for the radial and angular parts which for most of the parameter space contain *more than 3* finite singularities and thereby generalise the Heun differential equations.
- We already mentioned regions of the parameter space (e.g. scalar mass) for which the FDE reduce to Heun equations.
- There are also other regions of the parameter space for which the extra singular points become *false* or *apparent* singular points.
- A singular point is called false if both of its exponents are non-negative integers and there are no logarithmic terms in the local expansion near the singular point.

- In our CQG paper, we derived a condition that guarantees the absence of logarithmic terms local to a singular point with exponents $(0, 2)$ and proved that for $\mu = \sqrt{\frac{5\Lambda}{12}}$, provided the coefficients of the angular equation satisfy the abovementioned condition, the extra singular point becomes a false point (G V Kraniotis 2016, Class. Quantum Grav. 33 225011).
- One might even go a step further, by conjecturing that in the case of a Fuchsian equation with 5 singular points, as it is for example the case of the radial part of the KGF equation for charged massive scalar particle in the KNdS background for most of the parameter space, that if one of the singular points is false than the solution will be expressed in terms of general Heun functions. The investigation of such a conjecture will be examined thoroughly.
- Additional ramifications to be explored include the Riemann-Hilbert problem which in layman's terms states: for a FDE in order that its isomonodromy problem is non-trivial additional degrees of freedom in the form of apparent singularities need to be introduced. If one considers the apparent singularities and the conjugate momenta as functions of the non-apparent singularities (other than 0 and 1) a system of Hamilton's equations emerges. For one additional false singularity this Hamiltonian system is equivalent to a non-linear second order ODE the celebrated sixth Painleve equation.

Exact solution of the radial equation for a massive charged scalar particle in the Kerr-Newman spacetime

The radial equation for $\Lambda = 0$ in this case is given by:

$$\Delta^{KN}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\Bigg[\omega^2(r^2+a^2)^2-4Ma\omega mr+2e^2a\omega m-\mu^2r^2\Delta^{KN}$$

$$+m^2a^2-(\omega^2a^2+\mathcal{K}_{lm})\Delta^{KN}-2eqr[(r^2+a^2)\omega-am]+e^2q^2r^2\Bigg]R=0 \quad (75)$$

Introducing a new independent variable χ through:

$$M\chi = r - r_+, \quad r_\pm = M \pm Md, \quad (76)$$

and $R(\chi) = Z(\chi)(\chi(\chi+2d))^{-1/2}$ the radial equation takes the form:

$$\frac{\mathrm{d}^2Z}{\mathrm{d}\chi^2}+\left[M^2(\omega^2-\mu^2)+\frac{1}{M^2}\left\{\frac{A'}{\chi^2}+\frac{B'}{\chi}+\frac{C'}{(\chi+2d)^2}+\frac{D'}{\chi+2d}\right\}\right]Z=0, \quad (77)$$

$$\begin{aligned}
A' &= A-\frac{1}{4d^2}\left(-e^2q^2M^2(1+d)^2+4eM^3q\omega(1+d)^2-2e^3qM\omega(d+1)\right)\\
B' &= B-\frac{1}{4d^3}\left(2aemMq+e^2M^2q^2(1-d^2)+2d^2M^4\mu^2(1+d)^2+4eM^3q\omega(d^3+2d^2-1)+2e^3qM\omega\right)\\
C' &= C-\frac{1}{4d^2}\left(2aeqmM(d-1)-e^2q^2M^2(1-d)^2+4eqM^3\omega(d-1)^2+2e^3qM\omega(d-1)\right)\\
D' &= D-\frac{1}{4d^3}(-2aeqmM-e^2q^2M^2(1-d^2)-2d^2M^4\mu^2(d-1)^2+4eq\omega M^3(1-2d^2+d^3)-2e^3qM\omega)
\end{aligned} \quad (78)$$

$$A = \frac{d^2M^2 + (am + (-2(1+d)M^2 + e^2)\omega)^2}{4d^2}, \tag{79}$$

$$B = \frac{1}{4d^3}(-a^2m^2 + d^2M^2(-1 - 2\mathcal{K}_{lm} - 2(1+d)^2M^2\mu^2) + 2am(2M^2 - e^2)\omega$$
$$- (2a^2d^2M^2 - 4(1+d)^2(-1+2d)M^4 + 4(-1+d^2)M^2e^2 + e^4)\omega^2) \tag{80}$$

$$C = \frac{d^2M^2 + (am + (2(-1+d)M^2 + e^2)\omega)^2}{4d^2} \tag{81}$$

$$D = \frac{1}{4d^3}(d^2M^2(1 + 2\mathcal{K}_{lm} + 2(-1+d)^2M^2\mu^2) + 2am(-2M^2 + e^2)\omega$$
$$+ (4(-1+d)^2(1+2d)M^4 + 4(-1+d^2)M^2e^2 + e^4)\omega^2 + a^2(m^2 + 2d^2M^2\omega^2)) \tag{82}$$

Using the change of variables

$$R(\chi) = e^{2i\zeta dM\sqrt{\omega^2-\mu^2}}\zeta^{\frac{\pm i}{2M}\sqrt{4A-M^2}}(\zeta-1)^{\frac{\pm i}{2M}\sqrt{4C-M^2}}Y(\zeta)\zeta^{1/2}(\zeta-1)^{1/2}(\chi(\chi+2d))^{-1/2} \tag{83}$$

yields the *confluent Heun equation*

$$\boxed{Y''(\zeta) + \left(\alpha + \frac{\gamma}{\zeta} + \frac{\delta}{\zeta-1}\right)Y'(\zeta) + \frac{w\zeta - \sigma}{\zeta(\zeta-1)}Y(\zeta) = 0} \tag{84}$$

(recall (17))

Exact solution of the radial equation for a massive charged scalar particle in the Kerr-Newman spacetime

A particular exact solution is

$$R(\zeta) = \frac{M}{\sqrt{\Delta^{KN}}} e^{-2idM\sqrt{\omega^2-\mu^2}\zeta} \zeta^{\frac{1}{2}-\frac{i}{2M}\sqrt{4A'-M^2}} (\zeta-1)^{\frac{1}{2}-\frac{i}{2M}\sqrt{4C'-M^2}} H_c(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta). \tag{85}$$

where

$$\alpha'_\pm = \pm 4idM\sqrt{\omega^2-\mu^2}, \quad \gamma'_\pm = 1 \pm \frac{i}{M}\sqrt{4A'-M^2}, \quad \delta'_\pm = 1 \pm \frac{i}{M}\sqrt{4C'-M^2}, \tag{86}$$

$$\begin{aligned} \sigma'_\pm = {} & \left(\frac{-2dB'}{M^2} - \frac{1}{2}\right) + \frac{1}{2} + \frac{\pm 4idM\sqrt{\omega^2-\mu^2}}{2}\left(1 \pm \frac{i}{M}\sqrt{4A'-M^2}\right) \\ & - \frac{1}{2}\left(1 \pm \frac{i}{M}\sqrt{4A'-M^2}\right)\left(1 \pm \frac{i}{M}\sqrt{4C'-M^2}\right) \end{aligned} \tag{87}$$

$$w'_\pm = \frac{-2d}{M^2}(B'+D') \pm 4idM\sqrt{\omega^2-\mu^2} \pm \frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left[\pm\frac{i}{M}\sqrt{4A'-M^2} \pm \frac{i}{M}\sqrt{4C'-M^2}\right], \tag{88}$$

and

$$\zeta = -\frac{\chi}{2d}. \tag{89}$$

Exact solution of the radial equation for a massive charged scalar particle in the Kerr-Newman spacetime

We can write the exact solution also in terms of the confluent Heun function, $\mathrm{HeunC}(\alpha_{\mathrm{M}}, \beta_{\mathrm{M}}, \gamma_{\mathrm{M}}, \delta_{\mathrm{M}}, \eta_{\mathrm{M}}, \zeta)$, defined in Maple. The correspondence among the parameters of the two functions H_c and HeunC is:

$$
\begin{aligned}
&Hc(\alpha'_{-}, w'_{-}, \gamma'_{-}, \delta'_{-}, \sigma'_{-}, \zeta) \to \mathrm{HeunC}\left(\alpha_{\mathrm{M}}, \beta_{\mathrm{M}}, \gamma_{\mathrm{M}}, \delta_{\mathrm{M}}, \eta_{\mathrm{M}}, \zeta\right) \\
&\alpha'_{-} = \alpha_{\mathrm{M}},\ \gamma'_{-} = 1+\beta_{\mathrm{M}},\ \delta'_{-} = 1+\gamma_{\mathrm{M}},\ \delta_{\mathrm{M}} = -\frac{2d}{M^2}(B'+D'),\ \eta_{\mathrm{M}} = \frac{1}{2} + \frac{2dB'}{M^2}, \\
&\sigma'_{-} = -\eta_{\mathrm{M}} + \frac{1}{2} + \frac{\alpha_{\mathrm{M}}}{2}(1+\beta_{\mathrm{M}}) - \frac{1}{2}(1+\beta_{\mathrm{M}})(1+\gamma_{\mathrm{M}}), \\
&w'_{-} = \delta_{\mathrm{M}} + \alpha_{\mathrm{M}} + \frac{\alpha_{\mathrm{M}}}{2}(\beta_{\mathrm{M}}+\gamma_{\mathrm{M}}).
\end{aligned}
\tag{90}
$$

A general solution for the radial equation of a charged massive particle in KN spacetime over the range $0 \leq \zeta < \infty$ can be written:

$$
\begin{aligned}
R(\zeta) = &\frac{M}{\sqrt{\Delta^{KN}}} e^{\frac{1}{2}\alpha_{\mathrm{M}}\zeta} \zeta^{\frac{1}{2}(1+\beta_{\mathrm{M}})} (\zeta-1)^{\frac{1}{2}(1+\gamma_{\mathrm{M}})} \{ c_1 \mathrm{HeunC}(\alpha_{\mathrm{M}}, \beta_{\mathrm{M}}, \gamma_{\mathrm{M}}, \delta_{\mathrm{M}}, \eta_{\mathrm{M}}, \zeta) \\
&+ c_2 \zeta^{-\beta_{\mathrm{M}}} \mathrm{HeunC}(\alpha_{\mathrm{M}}, -\beta_{\mathrm{M}}, \gamma_{\mathrm{M}}, \delta_{\mathrm{M}}, \eta_{\mathrm{M}}, \zeta) \}.
\end{aligned}
\tag{91}
$$

with c_1, c_2 constants.

The near event horizon limit-$\chi \rightarrow 0 \Leftrightarrow r \rightarrow r_+$

By $\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} = \left[M^2(\mu^2-\omega^2) - \frac{C'}{4M^2d^2} - \frac{D'}{2M^2 d} - \frac{1}{M^2}\left(\frac{A'}{\chi^2}+\frac{B'}{\chi}\right) + \mathcal{O}(\chi)\right] Z$ we derive for small χ-and neglecting terms of $\mathcal{O}(\chi)$-a Whittaker's differential equation:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\eta'^2} = \left[\frac{1}{4} - \frac{1}{M^2}\left(\frac{A'}{\eta'^2} + \frac{B'}{2\sqrt{\mathcal{F}'}}\frac{1}{\eta'}\right)\right] Z \tag{92}$$

where

$$\mathcal{F}' = M^2(\mu^2-\omega^2) - \frac{C'}{4M^2d^2} - \frac{D'}{2M^2 d}, \quad \eta' := 2\sqrt{\mathcal{F}'}\chi \tag{93}$$

and with the parameters $k' := \frac{B'}{2M^2\sqrt{\mathcal{F}'}}$, $m_h'^2 := 1/4 - \frac{A'}{M^2}$. Near the event horizon limit and expanding Kummer's confluent hypergeometric function which is involved in the solution of Whittaker's equation we find:

$$\begin{aligned} M_{k',m_h'}(\eta') &= e^{-\frac{\eta'}{2}} \eta'^{\frac{1}{2}+m_h'} F(m_h' + \frac{1}{2} - k', 2m_h'+1, \eta') \\ &= e^{-\frac{\eta'}{2}} \eta'^{\frac{1}{2}+m_h'} \sum_{\nu=0}^{\infty} \frac{(m_h'+\frac{1}{2}-k')_\nu}{(2m_h'+1)_\nu} \frac{\eta'^\nu}{\nu!} \end{aligned} \tag{94}$$

$$R(r) \sim \begin{cases} \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}} \left(2\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}\right)^{\frac{1}{2}+m'_h}, & r \to r_+ \\ \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}} \left(2\sqrt{\mathcal{F}'}\frac{(r-r_+)}{M}\right)^{\frac{1}{2}-m'_h}, & r \to r_+ \end{cases} \tag{95}$$

We can also use the convergent power series for the confluent Heun function in the vicinity of $\zeta = 0$ to derive the near event horizon limit:

$$H_c(\alpha', w', \gamma', \delta', \sigma', \zeta) = \sum_{k=0}^{\infty} c_k \zeta^k = 1 + \frac{\sigma'}{-\gamma'}\zeta + \frac{-(-\alpha' + \gamma' + \delta')\sigma' + \sigma'^2 + w'\gamma'}{2\gamma'(1+\gamma')}\zeta^2 + \cdots \tag{96}$$

$$R(r) \sim \begin{cases} \frac{M}{\sqrt{\Delta^{KN}}} \left[-\frac{(r-r_+)}{2dM}\right]^{\frac{1}{2}-\frac{i}{2M}\sqrt{4A'-M^2}} \\ \frac{M}{\sqrt{\Delta^{KN}}} \left[-\frac{(r-r_+)}{2dM}\right]^{\frac{1}{2}+\frac{i}{2M}\sqrt{4A'-M^2}} \end{cases} \tag{97}$$

When we compare our results in (97) with the result in (95) and by expanding the latter equation up to the first order in $\chi = (r - r_+)/M$, we see that the two results agree, except for a multiplicative constant.

Asymptotic solutions at infinity-$r \to \infty$

Using expansions, for large χ,

$$\frac{C'}{(\chi+2d)^2} = C'\left[\frac{1}{\chi^2} - \frac{4d}{\chi^3} + \cdots\right], \tag{98}$$

$$\frac{D'}{\chi+2d} = D'\left[\frac{1}{\chi} - \frac{2d}{\chi^2} + \cdots\right] \tag{99}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\chi^2} = \left[M^2(\mu^2-\omega^2) - \frac{1}{M^2}\left(\frac{A'+C'-2dD'}{\chi^2} + \frac{B'+D'}{\chi}\right) + \mathcal{O}\left(\frac{1}{\chi^3}\right)\right] Z$$

Introducing the variable $\xi = 2M(\mu^2-\omega^2)^{1/2}\chi$ in the large χ limit the radial differential equation reduces to the Whittaker differential equation:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\xi^2} = \left(\frac{1}{4} - \frac{k}{\xi} + \frac{m^2-\frac{1}{4}}{\xi^2}\right) Z, \tag{100}$$

$$k = \frac{B'+D'}{2M^3\sqrt{\mu^2-\omega^2}}, \quad \frac{1}{4} - m^2 = \frac{1}{M^2}[A'+C'-2dD']. \tag{101}$$

The solutions then far from the event horizon will involve the Kummer and Tricomi confluent hypergeometric functions. Indeed, standard solutions of (100) are

$$M_{k,m}(\xi) = e^{-\xi/2}\xi^{m+1/2}M(m-k+\frac{1}{2}, 2m+1, \xi), \tag{102}$$

$$W_{k,m}(\xi) = e^{-\xi/2}\xi^{m+1/2}U(m-k+\frac{1}{2}, 2m+1, \xi). \tag{103}$$

Using the important asymptotic series of the Tricomi function $(b = 1 + a - c)$ (Olver F W J, 1974):

$$\begin{aligned} U(a, c, \xi) &\sim \xi^{-a}\left[1 - \frac{ab}{\xi} + \frac{a(a+1)b(b+1)}{2!\xi^2} - \cdots\right] \\ &= \frac{1}{\xi^a}\sum_{\nu=0}^{\infty}\frac{(a)_\nu (b)_\nu}{(1)_\nu}\left(\frac{-1}{\xi}\right)^{\nu} \quad \xi \text{ large} \end{aligned} \tag{104}$$

yields

$$W_{k,m}(\xi) \sim e^{-\xi/2}\xi^k \left[1 + \frac{m^2 - \left(k-\frac{1}{2}\right)^2}{\xi} + \frac{\left(m^2-\left(k-\frac{1}{2}\right)^2\right)\left(m^2-\left(k-\frac{3}{2}\right)^2\right)}{2!\xi^2} + \cdots\right] \tag{105}$$

Thus we find as $\chi \to \infty \Leftrightarrow r \to \infty$ the solutions of the radial KGF equation for a massive charged particle are:

$$R(r) \sim \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mu^2-\omega^2}(r-r_+)}[2(\mu^2-\omega^2)^{1/2}(r-r_+)]^{\frac{B'+D'}{2M^3\sqrt{\mu^2-\omega^2}}} \tag{106}$$

where the particles electric charge contribution is through the factors B', D'. Since $W_{-k,m}(-\xi)$ forms another independent solution of the Whittaker equation we also have that as $r \to \infty$

$$R(r) \sim \frac{M}{\sqrt{\Delta^{KN}}} e^{+\sqrt{\mu^2-\omega^2}(r-r_+)}[-2(\mu^2-\omega^2)^{1/2}(r-r_+)]^{-\frac{B'+D'}{2M^3\sqrt{\mu^2-\omega^2}}} \tag{107}$$

Asymptotic solutions at infinity-$r \to \infty$ from asymptotics of Heun functions

We can also obtain the far horizon limit of our closed form analytic radial solutions as follows. The CHE (84) is a differential equation with an irregular singularity at infinity. Thus following the discovery of Thomé that such a differential equation can be satisfied in the neighbourhood of an irregular singularity by a series of the form

$$Y = e^{\lambda\zeta}\zeta^{\mu}\sum_{s=0}^{\infty}\frac{a_s}{\zeta^s} \tag{108}$$

we determine the exponential parameters λ, μ for the case of CHE to be:

$$\lambda_1 = 0, \qquad \mu_1 = -\frac{w'_-}{\alpha'_-} = -\left[\frac{\delta_{\mathrm{M}}}{\alpha_{\mathrm{M}}} + \frac{1}{2}(2+\beta_{\mathrm{M}}+\gamma_{\mathrm{M}})\right] \tag{109}$$

$$\lambda_2 = -\alpha'_- = -\alpha_{\mathrm{M}}, \quad \mu_2 = \left[-(\gamma'_- + \delta'_-) + \frac{w'_-}{\alpha'_-}\right] = \frac{\delta_{\mathrm{M}}}{\alpha_{\mathrm{M}}} - \frac{1}{2}(2+\beta_{\mathrm{M}}+\gamma_{\mathrm{M}}) \tag{110}$$

Thus for $r \to \infty$

$$H_c(\alpha'_-, w'_-, \gamma'_-, \delta'_-, \sigma'_-, \zeta) \sim \begin{cases} \zeta^{-\frac{w'_-}{\alpha'_-}} \\ e^{-\alpha'_-\zeta}\zeta^{-(\gamma'_- + \delta'_-) + \frac{w'_-}{\alpha'_-}} \end{cases} \Leftrightarrow$$

$$\mathrm{HeunC}(\alpha_{\mathrm{M}}, \beta_{\mathrm{M}}, \gamma_{\mathrm{M}}, \delta_{\mathrm{M}}, \eta_{\mathrm{M}}, \zeta) \sim \begin{cases} \zeta^{-\left[\frac{\delta_{\mathrm{M}}}{\alpha_{\mathrm{M}}} + \frac{1}{2}(2+\beta_{\mathrm{M}}+\gamma_{\mathrm{M}})\right]} \\ e^{-\alpha_{\mathrm{M}}\zeta}\zeta^{\frac{\delta_{\mathrm{M}}}{\alpha_{\mathrm{M}}} - \frac{1}{2}(2+\beta_{\mathrm{M}}+\gamma_{\mathrm{M}})} \end{cases}$$

Thus in terms of the original variables we find:

$$R(r) \sim \begin{cases} \frac{M}{\sqrt{\Delta^{KN}}} e^{-\sqrt{\mu^2-\omega^2}(r-r_+)} \left(-\frac{r-r_+}{2Md}\right)^{\frac{(B'+D')}{2M^3\sqrt{\mu^2-\omega^2}}} \\ \frac{M}{\sqrt{\Delta^{KN}}} e^{+\sqrt{\mu^2-\omega^2}(r-r_+)} \left(-\frac{r-r_+}{2Md}\right)^{-\frac{(B'+D')}{2M^3\sqrt{\mu^2-\omega^2}}} \end{cases} \tag{111}$$

When we compare the results in (111) with the results in (106) and (107) we see they differ slightly. The are equivalent, except for a multiplicative constant.

Possible applications:superradiance instabilities of a KN(a)dS black hole

- Possible application of our analytic exact solutions: gravitational radiation from a hypothetical axion cloud around a rotating charged BH. Ultralight axion fields are ubiquitous in Calabi-Yau compactifications of string theory.
- Rotating bodies amplify incident waves-*superradiance* (Ya B. Zel'dovich, JETP 1972).
- Superradiant instability (SI) takes place when the Compton wavelength of the axion mass has the order of the BH gravitational radius. GW emissions from a hypothetical axion bosonic cloud around the galactic centre supermassive BH SgrA* can constrain the mass of ultralight axions (of order 10^{-16}eV) (Arvanitaki *et al* 2010, Yoshino & Kodama 2014).
- Extending our results, by constructing radial solutions valid in all regions of r, we will be able to look for possible signature of the *static radius* in the behaviour of the scalar massive field in the KdS and KNdS spacetimes. (Static radius separates in KdS spacetime the region that corresponds to gravitational binding with that of cosmic repulsion where gravitational binding is not possible-Z. Stuchlík 2005).
- Further interesting research investigations include superradiance for bosonic massive fields with spin and/or fermionic massive degrees of freedom.
- Studies of this kind aided by precise measurements of relativistic effects for the SgrA* BH will eventually offer the exciting possibilty of testing the BH "no hair" hypothesis.

Conclusions

- Exact analytic solutions for the massive KFG equation for a charged particle in the curved spacetime of a KNdS black hole were derived.
- We first proved the separation of massive KGF equation in the KN-(a)dS black hole spacetime. This separation of variables yielded the corresponding radial and angular Fuchsian differential equations.
- The resulting Fuchsian differential eqns. in the general case contain more than 4 regular singularities-thus they suggest a generalisation of Heun functions.
- For the value of the inverse Compton wavelength of the scalar particle, $\mu = \sqrt{\frac{2\Lambda}{3}}$, both radial and angular parts of the separated KGF equation were transformed into Heun equations. We then solved both Heun equations in terms of an infinite series of hypergeometric functions using the idea of augmented convergence.
- We investigated false singularities. We derived, a condition that guarantees the absence of logarithmic terms local to a singular point with exponents $(0, 2)$ and proved that for $\mu = \sqrt{\frac{5}{12}\Lambda}$, provided the coefficients of the angular equations satisfy the mentioned condition, the extra singular point becomes a false point.
- The closed form analytic solutions for the radial and angular equations for a massive charged scalar particle in the KN spacetime are expressed in terms of *confluent Heun functions*. Asymptotics were investigated.
- Studies of this kind will eventually offer the exciting possibility of testing the BH "no hair" hypothesis.