All-loop non-Abelian Thirring model

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based on works with
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Introduction and motivation

Exact β-functions and anomalous dimensions

1. In a renormalizable field theory, its quantum behaviour is depicted by:
   - The n-point correlation functions.
   - The dependence of the coupling with the energy scale.

2. Their dependence is encoded within the RG flow equations

$$\beta = \frac{d\lambda}{d\ln\mu^2}\,,$$

   which are usually determined perturbatively.

3. Can we obtain the all-loop β-function? New fixed points towards the IR?

4. Can we also calculate the all-loop correlators of various operators?

We study the above in the non-Abelian bosonized Thirring model.

FOCAL POINTS

- Non-Abelian Thirring model
- The effective action
- Its β-function
- Current correlators in the non-Abelian Thirring model
- OPEs & equal-time commutators
- Conclusion & Outlook

Plan of the talk

1. Non-Abelian Thirring model
2. The effective action
3. The β-function
4. Current correlators
5. Conclusion & Outlook

FERMIONIC MODEL

Exactly solvable QFT describing self-interacting massless Dirac fields in 1+1 dimensions.

- An 1+1 dimensional action with fermions in the fundamental representation of $SU(N)$

$$\text{Dashen–Frishman (1973) \& (1975)}: \quad \mathcal{L}_{int} = -\frac{g_B}{2} J_\mu J^\mu - \frac{g_V}{2} J^a_\mu J^{a\mu}\,, \quad \mu = 0, 1\,,$$

- It is invariant under $SU(N) \times U(1)_{\text{Vector}}$ and $U(1)_{Axial}$
- $J^a_\mu = \bar{\Psi} t^a \gamma_\mu \Psi$ the $SU(N)$ currents, $J_\mu = \bar{\Psi}\gamma_\mu\Psi$ the $U(1)_{\text{Vector}}$ and $J^5_\mu = \varepsilon_{\mu\nu} J^\nu$ the $U(1)_{Axial}$
- The non-Abelian term breaks $SU(N)_{Axial}$, i.e. $\partial^\mu J^{5a}_\mu = g_V f_{abc} J^{b\mu} J^{5c}_\mu$
- For $N = 1$ we recover the Abelian case (prototype) Thirring (1958)
- The theory is scale-invariant only for $g_V = 0$ and $g_V = \frac{4\pi}{n+1}$
- There is a current algebra at *level one*:

$$J^a_\pm(z) J^b_\pm(0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc} J^c_\pm(0)}{z}$$

Non-Abelian Thirring model

Consider the WZW action Witten (1983)

$$S_{\text{WZW},k}(g) = -\frac{k}{2\pi}\int d^2\sigma\, \text{Tr}\left(g^{-1}\partial_+ g\, g^{-1}\partial_- g\right) + \frac{k}{12\pi}\int_B \text{Tr}\left(g^{-1}dg\right)^3 ,$$

invariant under the left-right current algebra symmetry: $g \mapsto \Omega^{-1}(\sigma_+)\, g\, \Omega(\sigma_-)$.

The holomorphic and anti-holomorphic currents obey the OPEs

$$J^a_\pm(z)J^b_\pm(0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc}\, J^c_\pm(0)}{\sqrt{k}\, z} + \text{regular}, \quad J^a_\pm(z)J^b_\mp(0) = \text{regular},$$

$$J^a_+ = -i\,\text{Tr}(t^a\, \partial_+ g\, g^{-1}), \quad J^a_- = -i\,\text{Tr}(t^a\, g^{-1}\, \partial_- g), \quad D_{ab} = \text{Tr}(t^a g t^b g^{-1}),$$

where: $[t_a, t_b] = f_{abc} t_c$, $\text{Tr}(t_a t_b) = \delta_{ab}$ and $f_{acd} f_{bcd} = -c_G\, \delta_{ab}$.

The non-abelian bosonized Thirring model is defined through

$$S = S_{\text{WZW},k} + k\,\frac{\lambda_{ab}}{\pi}\int d^2\sigma\, J^a_+\, J^b_-$$

Non-Abelian Thirring model

Symmetries of the non-abelian bosonized Thirring model

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^b$$

1. It is invariant under the generalized parity symmetry

$$\lambda \mapsto \lambda^T\,, \qquad g \mapsto g^{-1}\,, \qquad \sigma^{\pm} \mapsto \sigma^{\mp}$$

2. The perturbation is not *exactly marginal*

Kutasov (1989) $$\beta = -\frac{c_G\,\lambda^2}{2k\,(1+\lambda)^2} \leqslant 0\,, \quad \lambda_{ab} = \lambda\,\delta_{ab}$$

3. The corresponding effective action is invariant under the inversion of the coupling:

Kutasov (1989) $$\lambda \mapsto \lambda^{-1}\,, \qquad k \mapsto -k\,, \qquad k \gg 1$$

4. The left-right current algebra symmetry is broken for a generic matrix λ_{ab}

Plan of the talk

The effective action

By a gauging procedure we can construct the following action Sfetsos (2013)

$$S_{k,\lambda}(g) = S_{\text{WZW},k} + \frac{k}{\pi}\int \mathrm{d}^2\sigma \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_+^a J_-^b\,, \quad 0 \leqslant \lambda < 1\,,$$

describes integrable interpolations from a WZW to (non-abelian T-duals) PCM models.
Sfetsos (2013), Itsios–Sfetsos–KS–Torrieli (2014)

Properties:

- For $\lambda \ll 1$ we get the non-Abelian Thirring model
- Invariance under the generalized parity symmetry $g \mapsto g^{-1}\,, \sigma^{\pm} \mapsto \sigma^{\mp}$
- Weak-strong duality $S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g)$
- It is integrable as the equations of motion can be written in terms of a Lax pair

$$\partial_{\pm} I_{\mp} = \mp\frac{1}{2}\left[I_+, I_-\right], \quad I_{\pm} = \frac{2}{1+\lambda} A_{\pm}\,,$$

$$A_+^a = \left(\lambda^{-1} - D\right)^{-1}_{ab} J_+^b\,, \quad A_-^a = -\left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_-^b$$

where

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ = \left[\mathcal{L}_+, \mathcal{L}_-\right], \quad \mathcal{L}_{\pm} = \frac{\mu}{\mu \mp 1} I_{\pm}\,, \quad \mu \in \mathbb{C}$$

LIMITING CASES

There are two interesting limits at $\lambda = \pm 1$:
Sfetsos (2013), Georgiou, Sfetsos, KS 2016

1. *Zoom around* $\lambda = 1$*:*

$$\lambda = 1 - \frac{\kappa^2}{k} + \dots\,, \quad g = \mathbb{I} + i\,\frac{v_a t^a}{k} + \dots\,, \quad k \gg 1\,,$$

we get the non-abelian T-dual

$$S_{\text{non-Abel}} = \frac{1}{\pi}\int \mathrm{d}^2\sigma\,\left(\kappa^2 + f\right)^{-1}_{ab}\partial_+ v^a \partial_- v^b\,, \quad f_{ab} := -i f_{abc}\, v^c\,,$$

of the principal chiral model (PCM) with respect to G_L or G_R

$$S_{\text{PCM}} = \frac{\kappa^2}{\pi}\int \mathrm{d}^2\sigma\,\mathrm{Tr}\left(g^{-1}\partial_+ g\, g^{-1}\partial_- g\right)$$

2. *Zoom around* $\lambda = -1$*:*

$$\lambda = -1 + \frac{1}{b^{2/3}\,k^{1/3}} + \dots\,, \quad g = \mathbb{I} + i\,\frac{v_a t^a}{k^{1/3}} + \dots\,, \quad k \gg 1\,,$$

we have the pseudodual model

$$S_{\text{pseudo-dual}} = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\left(\frac{\delta_{ab}}{b^{2/3}} + \frac{1}{3}f_{ab}\right)\partial_+ v^a \partial_- v^b$$

Nappi 1980

Plan of the talk

CONSTRAINTS ON THE β-FUNCTION

The β-function at one-loop in $1/k$ expansion takes the form:

$$\beta = \frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu^2} = -\frac{1}{k} f(\lambda)\,.$$

- From CFT perturbations we expect that:

$$f(\lambda) \simeq \frac{1}{2}\, c_G\, \lambda^2 + \mathcal{O}(\lambda^3)$$

- Due to the weak–strong duality we have the constraint:

$$f(\lambda) = \lambda^2 f(\lambda^{-1})\,.$$

- Let's compute $f(\lambda)$ throughout the effective action.

GENERAL APPROACH

Consider a 1+1-dimensional non-linear σ-model with action

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma\, E_{\mu\nu}\, \partial_+ X^\mu \partial_- X^\nu\,, \quad E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

The one-loop β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:
Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85

$$\frac{\mathrm{d}\, E_{\mu\nu}}{\mathrm{d}\ln\mu^2} = R^-_{\mu\nu} + \nabla^+_\nu \xi_\mu\,,$$

where the last term corresponds to field redefinitions (diffeomorphisms).

Generalities:

- The Ricci tensor and the covariant derivative includes torsion, i.e. $H = \mathrm{d}B$
- The σ-model is renormalizable within the zoo of metrics and 2-forms
- Not given that the RG flows will retain the form at hand of $G_{\mu\nu}$ and $B_{\mu\nu}$

Isotropic case

The RG flow at one-loop in $1/k$ expansion retains the form of the σ-model

$$\text{Itsios–Sfetsos–KS (2014)} \quad \beta = \frac{d\lambda}{d\ln\mu^2} = -\frac{c_G\,\lambda^2}{2k(1+\lambda)^2}\,, \quad 0 \leqslant \lambda < 1\,, \quad k \text{ does not flow}$$

Properties of the flow:

1. In agreement with the all-loop isotropic Thirring model Kutasov 89
2. Invariance under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k \quad \text{for} \quad k \gg 1$
3. It behaves according to CFT expectations around $\lambda \ll 1 \Longrightarrow \beta \simeq -\frac{c_G\,\lambda^2}{2k} + \mathcal{O}(\lambda^3)$
4. The β-function can be solved explicitly:

$$\lambda - \lambda^{-1} + 2\ln\lambda = -\frac{c_G}{2k}\,(t - t_0)\,, \quad t = \ln\mu^2,$$

where UV at $\lambda \to 0^+$ and towards the IR at $\lambda \to 1^-$

Claim

The effective action incorporates the all-loop counterterms of the non-abelian bosonized Thirring model

Plan of the talk

β-FUNCTION AND ANOMALOUS DIMENSION

Starting point is the non-Abelian Thrirring model action for isotropic deformations

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda}{\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^a\,.$$

Perturbative results, well defined behaviour at $\lambda = \pm 1$ and weak-strong duality:
See Sfetsos talk

$$\beta = -\frac{c_G\,\lambda^2}{2k\,(1+\lambda)^2} \leqslant 0\,, \qquad \gamma^{(J)} = \frac{c_G\,\lambda^2}{k\,(1-\lambda)(1+\lambda)^3} \geqslant 0$$

Itsios, Sfetsos, KS (2014), Appadu, Hollowood (2015), Georgiou, Sfetsos, KS (2015)

CORRELATORS AND OPE

All-loop two and three-point functions – leading in $1/\sqrt{k}$ expansion

$$\langle J^a(x_1)J^b(x_2)\rangle_{k,\lambda} = \frac{\delta_{ab}}{x_{12}^2}\,, \quad \langle J^a(x_1)\bar{J}^b(\bar{x}_2)\rangle_{k,\lambda} = 0\,,$$

$$\langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle_{k,\lambda} = \frac{1+\lambda+\lambda^2}{\sqrt{k(1-\lambda)(1+\lambda)^3}}\,\frac{f_{abc}}{x_{12}x_{13}x_{23}}\,,$$

$$\langle J^a(x_1)J^b(x_2)\bar{J}^c(\bar{x}_3)\rangle_{k,\lambda} = \frac{\lambda}{\sqrt{k(1-\lambda)(1+\lambda)^3}}\,\frac{f_{abc}\bar{x}_{12}}{x_{12}^2\bar{x}_{13}\bar{x}_{23}}\,,$$

where $x := \sigma + i\,\tau$ and $x_{12} := x_1 - x_2$.
Georgiou, Sfetsos, KS (2016), Konechny, Quella (2011) for supergroups.

Using the above we find the operator product expansion (OPE) algebra

$$J^a(x_1)J^b(x_2) = \frac{\delta_{ab}}{x_{12}^2} + \frac{\mathrm{e}}{2}\,(1+2\,\chi)\,\frac{f_{abc}J^c(x_2)}{x_{12}} + \frac{\mathrm{e}}{2}\,\frac{f_{abc}\bar{J}^c(\bar{x}_2)\bar{x}_{12}}{x_{12}^2} + \dots\,,$$

$$J^a(x_1)\bar{J}^b(\bar{x}_2) = \frac{\mathrm{e}}{2}\,\frac{f_{abc}\bar{J}^c(\bar{x}_2)}{x_{12}} + \frac{\mathrm{e}}{2}\,\frac{f_{abc}J^c(x_2)}{\bar{x}_{12}} + \dots\,,$$

where:

$$\mathrm{e} = \frac{1}{\sqrt{k(1-\lambda^2)}}\,\frac{2\lambda}{1+\lambda}\,, \quad \chi = \frac{1+\lambda^2}{2\lambda}\,.$$

CURRENT ALGEBRA

Having the OPEs, we can compute the equal-time commutators:

$$
\begin{aligned}
[S^a(\sigma_1), S^b(\sigma_2)] &= \frac{i\,k}{2\pi}\,\delta_{ab}\delta'(\sigma_{12}) + f_{abc}\,S^c(\sigma_2)\,\delta(\sigma_{12})\,,\\
[\bar{S}^a(\sigma_1), \bar{S}^b(\sigma_2)] &= -\frac{i\,k}{2\pi}\,\delta_{ab}\delta'(\sigma_{12}) + f_{abc}\,\bar{S}^c(\sigma_2)\,\delta(\sigma_{12})\,,\\
[S^a(\sigma_1), \bar{S}^b(\sigma_2)] &= 0\,,
\end{aligned}
$$

with

$$
S^a = \frac{1}{2\pi}\sqrt{\frac{k}{1-\lambda^2}}\,\left(J^a - \lambda\,\bar{J}^a\right)\,, \quad \bar{S}^a = \frac{1}{2\pi}\sqrt{\frac{k}{1-\lambda^2}}\,\left(\bar{J}^a - \lambda\,J^a\right)\,.
$$

Rajeev (1989), Georgiou, Sfetsos, KS (2016)

Comments:

1. Two-commuting λ-independent current algebras
2. The effective Hamiltonian expressed in terms of $(S, \bar{S})$ depends on the deformation parameter λ

Comments on the current algebra

1. Rajeev's one-parameter deformation of the canonical structure of the isotropic PCM.
2. Taking its classical limit and appropriately rescaling $(J^a, \bar{J}^a)$ we find:

$$\{I^a_\pm, I^b_\pm\}_{\mathrm{P.B.}} = -i\, \mathrm{e}^2 f_{abc}\left(I^c_\mp(\sigma_2) - (1+2x) I^c_\pm(\sigma_2)\right)\delta(\sigma_{12}) \pm 2\mathrm{e}^2\delta_{ab}\,\delta'(\sigma_{12})\,,$$
$$\{I^a_\pm, I^b_\mp\}_{\mathrm{P.B.}} = i\,\mathrm{e}^2 f_{abc}\left(I^c_+(\sigma_2) + I^c_-(\sigma_2)\right)\delta(\sigma_{12})\,,$$

   recall that f_{abc} are taken to be imaginary.
3. This is the underlying structure of the integrable λ-deformed model, with:

$$I_\pm = \frac{2}{1+\lambda} A_\pm\,, \quad A^a_+ = \left(\lambda^{-1} - D\right)^{-1}_{ab} J^b_+\,, \quad A^a_- = -\left(\lambda^{-1} - D^T\right)^{-1}_{ab} J^b_-$$

Result

The effective action

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k} + \frac{k}{\pi}\int \mathrm{d}^2\sigma\,\left(\lambda^{-1} - D^T\right)^{-1}_{ab} J^a_+ J^b_-$$

incorporates the all-loop counterterms of the non-abelian bosonized Thirring model

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda}{\pi}\int \mathrm{d}^2\sigma\, J^a_+ J^a_-$$

Plan of the talk

Conclusion & Outlook

Our resumed action

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k} + \frac{k}{\pi}\int \mathrm{d}^2\sigma\, \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_+^a J_-^b$$

enraptures the all-loop isotropic Thirring model at leading order in $1/k$ expansion

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda}{\pi}\int \mathrm{d}^2\sigma\, J_+^a J_-^a$$

The agreement is based upon:

1. Symmetries of the actions
2. Invariance under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k \quad \text{for} \quad k \gg 1$
3. β-function and the anomalous dimension of the currents $\gamma^{(J)}$
4. Rajeev's one-parameter deformation of the PB of the isotropic PCM

Extensions:

- Cases beyond isotropy $\lambda_{ab} \neq \lambda\,\delta_{ab}$
- We can also include affine primary fields and introduce different current levels $k_{L,R}$ Georgiou, Sfetsos, KS (2016)
- Subleading in $1/k$ expansion, beyond the weak–strong duality: Kutasov (1989) $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k - 2c_G$