All-loop non-Abelian Thirring model

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based on works with
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INRODUCTION AND MOTIVATION

Exact $\beta$-functions and anomalous dimensions

1. In a renormalizable field theory, its quantum behaviour is depicted by:
   - The $n$-point correlation functions.
   - The dependence of the coupling with the energy scale.

2. Their dependence is encoded within the RG flow equations

$$\beta = \frac{d\lambda}{d \ln \mu^2},$$

which are usually determined perturbatively.

3. Can we obtain the all-loop $\beta$-function? New fixed points towards the IR?

4. Can we also calculate the all-loop correlators of various operators?

We study the above in the non-Abelian bosonized Thirring model.
Focal Points

- Non-Abelian Thirring model
- The effective action
- Its $\beta$-function
- Current correlators in the non-Abelian Thirring model
- OPEs & equal-time commutators
- Conclusion & Outlook
Plan of the talk

1. Non-Abelian Thirring model
2. The effective action
3. The $\beta$-function
4. Current correlators
5. Conclusion & Outlook
**Fermion model**

Exactly solvable QFT describing self-interacting massless Dirac fields in 1+1 dimensions.

- An 1+1 dimensional action with fermions in the fundamental representation of $SU(N)$

\[ \mathcal{L}_{\text{int}} = -\frac{g_B}{2} J^\mu J^\mu - \frac{g_V}{2} J^a_{\mu} J^{a\mu}, \quad \mu = 0, 1, \]

- It is invariant under $SU(N) \times U(1)_{\text{Vector}}$ and $U(1)_{\text{Axial}}$

- $J^a_\mu = \bar{\Psi} t^a \gamma_\mu \Psi$ the $SU(N)$ currents, $J_\mu = \bar{\Psi} \gamma_\mu \Psi$ the $U(1)_{\text{Vector}}$ and $J^5_\mu = \varepsilon_{\mu \nu} J^\nu$ the $U(1)_{\text{Axial}}$

- The non-Abelian term breaks $SU(N)_{\text{Axial}}$, i.e. $\partial^\mu J^5_\mu = g_V f_{abc} J^b_{\mu} J^c_{\mu}$

- For $N = 1$ we recover the Abelian case (prototype) Thirring (1958)

- The theory is scale-invariant only for $g_V = 0$ and $g_V = \frac{4\pi}{n+1}$

- There is a current algebra at level one:

\[ J^a_\pm (z) J^b_\pm (0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc} J^c_\pm (0)}{z} \]
**Non-Abelian Thirring model**

Consider the WZW action 
\[ S_{WZW,k}(g) = -\frac{k}{2\pi} \int d^2 \sigma \text{Tr} \left(g^{-1} \partial_+ g g^{-1} \partial_- g\right) + \frac{k}{12\pi} \int_B \text{Tr} \left(g^{-1} dg\right)^3, \]
invariant under the left-right current algebra symmetry: 
\[ g \mapsto \Omega^{-1}(\sigma_+) g \Omega(\sigma_-). \]

The holomorphic and anti-holomorphic currents obey the OPEs
\[ J^a_\pm(z) J^b_\mp(0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc} J^c_\pm(0)}{\sqrt{k} z} + \text{regular}, \quad J^a_\pm(z) J^b_\mp(0) = \text{regular}, \]
\[ J^a_+ = -i \text{Tr}(t^a \partial_+ g g^{-1}), \quad J^a_- = -i \text{Tr}(t^a g^{-1} \partial_- g), \quad D_{ab} = \text{Tr}(t^a g t^b g^{-1}), \]

where: 
\[ [t_a, t_b] = f_{abc} t_c, \quad \text{Tr}(t_a t_b) = \delta_{ab} \text{ and } f_{acd} f_{bcd} = -c_G \delta_{ab}. \]

The non-abelian bosonized Thirring model is defined through
\[ S = S_{WZW,k} + k \frac{\lambda_{ab}}{\pi} \int d^2 \sigma J^a_+ J^b_- \]
NON-ABELIAN THIRRING MODEL

Symmetries of the non-abelian bosonized Thirring model

\[ S = S_{WZW,k} + k \frac{\lambda_{ab}}{\pi} \int d^2 \sigma J^a_+ J^b_- \]

1. It is invariant under the generalized parity symmetry

\[ \lambda \mapsto \lambda^T, \quad g \mapsto g^{-1}, \quad \sigma^\pm \mapsto \sigma^{\mp} \]

2. The perturbation is not exactly marginal

Kutasov (1989) \[ \beta = -\frac{c_G \lambda^2}{2k (1 + \lambda)^2} \leq 0, \quad \lambda_{ab} = \lambda \delta_{ab} \]

3. The corresponding effective action is invariant under the inversion of the coupling:

Kutasov (1989) \[ \lambda \mapsto \lambda^{-1}, \quad k \mapsto -k, \quad k \gg 1 \]

4. The left-right current algebra symmetry is broken for a generic matrix \( \lambda_{ab} \)
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The effective action

By a gauging procedure we can construct the following action Sfetsos (2013)

\[ S_{k,\lambda}(g) = S_{WZW,k} + \frac{k}{\pi} \int d^2 \sigma \left( \lambda^{-1} - D^T \right)^{-1}_{ab} J^a_+ J^b_-, \quad 0 \leq \lambda < 1, \]


Properties:

- For \( \lambda \ll 1 \) we get the non-Abelian Thirring model
- Invariance under the generalized parity symmetry \( g \mapsto g^{-1}, \sigma^\pm \mapsto \sigma^{\mp} \)
- Weak-strong duality \( S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g) \)
- It is integrable as the equations of motion can be written in terms of a Lax pair

\[
\partial_\pm I_\mp = \mp \frac{1}{2} [I_+, I_-], \quad I_\pm = \frac{2}{1 + \lambda} A_\pm, \\
A_+^a = \left( \lambda^{-1} - D \right)^{-1}_{ab} J_+^b, \quad A_-^a = - \left( \lambda^{-1} - D^T \right)^{-1}_{ab} J_-^b
\]

where

\[
\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ = [\mathcal{L}_+, \mathcal{L}_-], \quad \mathcal{L}_\pm = \frac{\mu}{\mu \mp 1} I_\pm, \quad \mu \in \mathbb{C}
\]
**Limiting Cases**

There are two interesting limits at $\lambda = \pm 1$:

Sfetsos (2013), Georgiou, Sfetsos, KS 2016

1. **Zoom around $\lambda = 1$:**

\[
\lambda = 1 - \frac{\kappa^2}{k} + \ldots, \quad g = \mathbb{1} + i \frac{v_{at}^a}{k} + \ldots, \quad k \gg 1,
\]

we get the non-abelian T-dual

\[
S_{\text{non-Abel}} = \frac{1}{\pi} \int d^2 \sigma \left( \kappa^2 + f \right)^{-1}_{ab} \partial^a v^b - f_{ab} \partial v^c,
\]

of the principal chiral model (PCM) with respect to $G_L$ or $G_R$

\[
S_{\text{PCM}} = \frac{\kappa^2}{\pi} \int d^2 \sigma \mathrm{Tr} \left( g^{-1} \partial g g^{-1} \partial g \right)
\]

2. **Zoom around $\lambda = -1$:**

\[
\lambda = -1 + \frac{1}{b^{2/3} k^{1/3}} + \ldots, \quad g = \mathbb{1} + i \frac{v_{at}^a}{k^{1/3}} + \ldots, \quad k \gg 1,
\]

we have the pseudodual model

\[
S_{\text{pseudo-dual}} = \frac{1}{4\pi} \int d^2 \sigma \left( \frac{\delta_{ab}}{b^{2/3}} + \frac{1}{3} f_{ab} \right) \partial^a v^b \partial v^b
\]

Nappi 1980
Plan of the talk

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**Constraints on the $\beta$-function**

The $\beta$-function at one-loop in $\frac{1}{k}$ expansion takes the form:

$$\beta = \frac{d\lambda}{d \ln \mu^2} = -\frac{1}{k} f(\lambda).$$

- From CFT perturbations we expect that:
  $$f(\lambda) \simeq \frac{1}{2} c_G \lambda^2 + \mathcal{O}(\lambda^3)$$

- Due to the weak–strong duality we have the constraint:
  $$f(\lambda) = \lambda^2 f(\lambda^{-1}).$$

- Let’s compute $f(\lambda)$ throughout the effective action.
**General Approach**

Consider a 1+1-dimensional non-linear $\sigma$-model with action

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma E_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu, \quad E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

The one-loop $\beta$-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:

Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85

$$\frac{d E_{\mu\nu}}{d \ln \mu^2} = R_{\mu\nu}^- + \nabla_+^\nu \xi_\mu,$$

where the last term corresponds to field redefinitions (diffeomorphisms).

Generalities:

- The Ricci tensor and the covariant derivative includes torsion, i.e. $H = dB$
- The $\sigma$-model is renormalizable within the zoo of metrics and 2-forms
- Not given that the RG flows will retain the form at hand of $G_{\mu\nu}$ and $B_{\mu\nu}$
**ISOTROPIC CASE**

The RG flow at one-loop in $1/k$ expansion retains the form of the $\sigma$-model

\[ \beta = \frac{d\lambda}{d \ln \mu^2} = -\frac{c_G \lambda^2}{2k(1+\lambda)^2}, \quad 0 \leq \lambda < 1, \quad k \text{ does not flow} \]

Properties of the flow:

1. In agreement with the all-loop isotropic Thirring model Kutasov 89
2. Invariance under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}, \quad k \mapsto -k \quad \text{for} \quad k \gg 1$
3. It behaves according to CFT expectations around $\lambda \ll 1 \implies \beta \sim -\frac{c_G \lambda^2}{2k} + O(\lambda^3)$
4. The $\beta$-function can be solved explicitly:

\[ \lambda - \lambda^{-1} + 2 \ln \lambda = -\frac{c_G}{2k} (t - t_0), \quad t = \ln \mu^2, \]

where UV at $\lambda \to 0^+$ and towards the IR at $\lambda \to 1^-$

**CLAIM**

The effective action incorporates the all-loop counterterms of the non-abelian bosonized Thirring model
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Starting point is the non-Abelian Thrirring model action for isotropic deformations

\[ S = S_{WZW,k} + k \frac{\lambda}{\pi} \int d^2 \sigma J_+^a J_-^a. \]

Perturbative results, well defined behaviour at \( \lambda = \pm 1 \) and weak-strong duality:
See Sfetsos talk

\[ \beta = -\frac{c_G \lambda^2}{2k (1 + \lambda)^2} \leq 0, \quad \gamma^{(J)} = \frac{c_G \lambda^2}{k (1 - \lambda)(1 + \lambda)^3} \geq 0 \]

CORRELATORS AND OPE

All-loop two and three-point functions – leading in $1/\sqrt{k}$ expansion

\[
\langle J^a(x_1)J^b(x_2) \rangle_{k,\lambda} = \frac{\delta_{ab}}{x_{12}^2}, \quad \langle J^a(x_1)\bar{J}^b(x_2) \rangle_{k,\lambda} = 0,
\]

\[
\langle J^a(x_1)J^b(x_2)J^c(x_3) \rangle_{k,\lambda} = \frac{1 + \lambda + \lambda^2}{\sqrt{k(1 - \lambda)(1 + \lambda)^3}} \frac{f_{abc}}{x_{12}x_{13}x_{23}},
\]

\[
\langle J^a(x_1)J^b(x_2)\bar{J}^c(x_3) \rangle_{k,\lambda} = \frac{\lambda}{\sqrt{k(1 - \lambda)(1 + \lambda)^3}} \frac{f_{abc}\bar{x}_{12}}{x_{12}^2x_{13}\bar{x}_{23}},
\]

where $x := \sigma + i\tau$ and $x_{12} := x_1 - x_2$.


Using the above we find the operator product expansion (OPE) algebra

\[
J^a(x_1)J^b(x_2) = \frac{\delta_{ab}}{x_{12}^2} + \frac{e}{2} (1 + 2\chi) \frac{f_{abc}J^c(x_2)}{x_{12}} + \frac{e}{2} \frac{f_{abc}\bar{J}^c(\bar{x}_2)\bar{x}_{12}}{x_{12}^2} + \ldots,
\]

\[
J^a(x_1)\bar{J}^b(\bar{x}_2) = \frac{e}{2} \frac{f_{abc}\bar{J}^c(\bar{x}_2)}{x_{12}} + \frac{e}{2} \frac{f_{abc}J^c(x_2)}{\bar{x}_{12}} + \ldots,
\]

where:

\[
e = \frac{1}{\sqrt{k(1 - \lambda^2)}} \frac{2\lambda}{1 + \lambda}, \quad \chi = \frac{1 + \lambda^2}{2\lambda}.
\]
CURRENT ALGEBRA

Having the OPEs, we can compute the equal-time commutators:

\[
[S^a(\sigma_1), S^b(\sigma_2)] = \frac{ik}{2\pi} \delta_{ab} \delta'(\sigma_{12}) + f_{abc} S^c(\sigma_2) \delta(\sigma_{12}) ,
\]

\[
[\bar{S}^a(\sigma_1), \bar{S}^b(\sigma_2)] = -\frac{ik}{2\pi} \delta_{ab} \delta'(\sigma_{12}) + f_{abc} \bar{S}^c(\sigma_2) \delta(\sigma_{12}) ,
\]

\[
[S^a(\sigma_1), \bar{S}^b(\sigma_2)] = 0 ,
\]

with

\[
S^a = \frac{1}{2\pi} \sqrt{\frac{k}{1 - \lambda^2}} \left( J^a - \lambda \bar{J}^a \right) , \quad \bar{S}^a = \frac{1}{2\pi} \sqrt{\frac{k}{1 - \lambda^2}} \left( \bar{J}^a - \lambda J^a \right) .
\]


Comments:

1. Two-commuting $\lambda$-independent current algebras

2. The effective Hamiltonian expressed in terms of $(S, \bar{S})$ depends on the deformation parameter $\lambda$
 COMMENTS ON THE CURRENT ALGEBRA

1. Rajeev’s one-parameter deformation of the canonical structure of the isotropic PCM.

2. Taking its classical limit and appropriately rescaling \((J^a, \bar{J}^a)\) we find:

\[
\{I^a_\pm, I^b_\pm\}_{\text{P.B.}} = -i e^2 f_{abc} (I^c_\pm (\sigma_2) - (1 + 2x)I^c_\pm (\sigma_2)) \delta(\sigma_{12}) \pm 2e^2 \delta_{ab} \delta'(\sigma_{12}),
\]

\[
\{I^a_\pm, I^b_\mp\}_{\text{P.B.}} = i e^2 f_{abc} (I^c_\pm (\sigma_2) + I^c_\pm (\sigma_2)) \delta(\sigma_{12}),
\]

recall that \(f_{abc}\) are taken to be imaginary.

3. This is the underlying structure of the integrable \(\lambda\)-deformed model, with:

\[
I_\pm = \frac{2}{1 + \lambda} A_\pm, \quad A^a_+ = \left(\lambda^{-1} - D\right)^{-1}_{ab} J^b_+, \quad A^a_- = -\left(\lambda^{-1} - D^T\right)^{-1}_{ab} J^b_-
\]

RESULT

The effective action

\[
S_{k,\lambda}(g) = S_{\text{WZW},k} + \frac{k}{\pi} \int d^2 \sigma \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J^a_+ J^b_-
\]

incorporates the all-loop counterterms of the non-abelian bosonized Thirring model

\[
S = S_{\text{WZW},k} + k \frac{\lambda}{\pi} \int d^2 \sigma J^a_+ J^a_-
\]
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CONCLUSION & OUTLOOK

Our resummed action

\[ S_{k,\lambda}(g) = S_{WZW,k} + \frac{k}{\pi} \int d^2\sigma \left( \lambda^{-1} - D^T \right)_{ab}^{-1} J^a_+ J^b_- \]

eraptures the all-loop isotropic Thirring model at leading order in \( 1/k \) expansion

\[ S = S_{WZW,k} + k \frac{\lambda}{\pi} \int d^2\sigma J^a_+ J^a_- \]

The agreement is based upon:

1. Symmetries of the actions
2. Invariance under the weak–strong duality, i.e. \( \lambda \mapsto \lambda^{-1}, \ k \mapsto -k \) for \( k \gg 1 \)
3. \( \beta \)-function and the anomalous dimension of the currents \( \gamma^{(J)} \)
4. Rajeev’s one-parameter deformation of the PB of the isotropic PCM

Extensions:

- Cases beyond isotropy \( \lambda_{ab} \neq \lambda \delta_{ab} \)
- We can also include affine primary fields and introduce different current levels \( k_L, k_R \)
  Georgiou, Sfetsos, KS (2016)
- Subleading in \( 1/k \) expansion, beyond the weak–strong duality:
  Kutasov (1989) \( \lambda \mapsto \lambda^{-1}, \ k \mapsto -k - 2c_G \)