THE GEOMETRY OF QUANTUM HALL EFFECT: AN EFFECTIVE ACTION FOR ALL DIMENSIONS

DIMITRA KARABALI

LEHMAN COLLEGE OF THE CUNY

Ioannina, HEP2017

APRIL 7, 2017
(2+1) dim QHE

- (2+1) dim systems of (nonrelativistic) electrons in strong magnetic field
- Hall conductivity is quantized

\[
J^i = \sigma_H \epsilon^{ij} E_j
\]
\[
\sigma_H = e^2 / h
\]

where \(\nu = \text{filling fraction}\)

\(\nu = 1, 2, \cdots\) for IQHE and \(\nu = 1/3, 1/5, \cdots\) for FQHE.

- Framework for interesting ideas
  - topological field theories (Chern-Simons effective actions)
  - bulk-edge dynamics
  - non-commutative geometries, fuzzy spaces
Charged particle moving on 2d plane (or $S^2$) in strong external magnetic field (Landau problem)

- Distinct Landau levels, separated by energy gap ($\sim B$)
- Each Landau level is degenerate
- Lowest Landau level (LLL):

$$\psi_n \sim z^n e^{-|z|^2/2}$$

$$z = x + iy$$
Quantum Hall Droplets

Many-body problem $\implies$ quantum Hall droplets

- Degeneracy of each LL is lifted by confining potential (for example: $V = \frac{1}{2} ur^2$)
- Exclusion principle $\implies$ N-body ground state = incompressible droplet
- Large $N \implies$ sharp boundary
Low energy excitations of droplets $\iff$ area preserving boundary fluctuations (edge excitations)

Edge dynamics is collectively described by 1d chiral boson $\phi$ \cite{Wen, Stone,...}

$$S_{edge} = \int_{\partial D} \left( \partial_t \phi + u \partial_\theta \phi \right) \partial_\theta \phi,$$

$$u \sim \left. \frac{\partial V}{\partial r^2} \right|_{\text{boundary}}$$
In the presence of electromagnetic interactions

\[ A_\mu = a_\mu + \delta A_\mu \]

The bulk dynamics is described by an effective action

\[ S_{\text{bulk}} = S_{\text{CS}} = \frac{\nu}{4\pi} \int_D \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \]

\( S_{\text{CS}} \) is not gauge invariant in presence of boundaries.

The edge dynamics is described by

\[ S_{\text{edge}} \sim \text{gauged chiral action} \]

Anomaly cancellation between bulk and edge actions,

\[ \delta S_{\text{bulk}} + \delta S_{\text{edge}} = 0 \]
The effective action $S_{CS}$ captures the response of the system to electromagnetic fluctuations

$$\frac{\delta S_{CS}}{\delta A_0} = \rho = \frac{\nu}{2\pi}$$

$$\frac{\delta S_{CS}}{\delta A_i} = J^i = \frac{\nu}{2\pi} \epsilon^{ij} E_j$$

Coefficient in front of $S_{CS}$ is related to Hall conductivity.

What about other transport coefficients?

- How does the system respond to stress and strain?
- Calculate stress tensor $\leftrightarrow$ couple theory to gravity
Effective action in background gauge field and metric

\[
S[\psi, \psi^\dagger; A_\mu, g_{ij}] = \int dtd^2 x \sqrt{g} \left[ i\hbar \psi^\dagger D_t \psi - \frac{\hbar^2}{2m} g^{ij} (D_i \psi)^\dagger (D_j \psi) \right]
\]

\[
\exp \left[ iS_{\text{eff}} \right] = \int d\psi \, d\psi^\dagger \exp \left[ iS[\psi, \psi^\dagger; A_\mu, g_{ij}] \right]
\]

\[
S_{\text{eff}} = \frac{1}{4\pi} \sum_{s=0}^{N-1} \int \left[ \left[ A + (s + \frac{1}{2}) \omega \right] d\left[ A + (s + \frac{1}{2}) \omega \right] - \frac{1}{12} \omega d\omega \right] + \cdots
\]

\[
\omega = \text{spin connection} \quad \text{s = 0 } \rightarrow \text{LLL, s = 1 } \rightarrow \text{1st LL, } \cdots
\]

\[
\frac{\delta S_{\text{eff}}}{\delta \omega_0} \sim n_H = \text{Hall viscosity}
\]

Abanov and Gromov, 2014
**Higher Dimensional QHE**

- **QHE on $S^4$** (Hu and Zhang)
  
  (background magnetic field = $SU(2)$ "instanton")

Generalization to arbitrary even (spatial) dimensions

- **QHE on $\mathbb{CP}^k$** (with V.P. Nair)

\[
\mathbb{CP}^k = \frac{SU(k+1)}{U(k)} \quad (2k \text{ dim space})
\]

$U(k) \sim U(1) \times SU(k) \implies$ We can have both $U(1)$ and $SU(k)$ background magnetic fields
QHE on a compact space $M \implies$ LLL defines an N-dim Hilbert space
In the presence of confining potential $\implies$ incompressible QH droplet

Density matrix for ground state droplet: $\hat{\rho}_0$

$$\hat{\rho}_0 = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}$$

$K$ filled states

Under time evolution: $\hat{\rho}_0 \rightarrow \hat{\rho} = \hat{U} \hat{\rho}_0 \hat{U}^\dagger$

$\hat{U} = N \times N$ unitary matrix; "collective" variable describing excitations within the LLL
The action for $\hat{U}$ is

$$S_0 = \int dt \ Tr \left[ i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} - \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right]$$

which leads to the evolution equation for density matrix

$$i\frac{d\hat{\rho}}{dt} = [\hat{V}, \hat{\rho}]$$

$S_0$ has no explicit dependence on properties of space on which QHE is defined, abelian or nonabelian nature of fermions, etc.
Noncommutative field theory

\[ S_0 = \text{action of a noncommutative field theory} \]

\[ S_0 = \int dt \, \text{Tr} \left[ i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} - \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right] \]

\[ = N \int d\mu \, dt \left[ i(\rho_0 * U^\dagger * \partial_t U) - (\rho_0 * U^\dagger * V * U) \right] \]

\[ \hat{\rho}_0, \hat{U}, \hat{V} \quad \Longrightarrow \quad \rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x}) \]

\((N \times N)\) matrices \quad symbols

- \(O(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}) \hat{O}_{ml}(t) \Psi_l^*(\vec{x})\)

- Matrix multiplication \quad \Longrightarrow \quad * \text{ product of symbols}

- \(\text{Tr} \quad \Longrightarrow \quad N \int d\mu\)

\(S_0 = \text{exact bosonic action describing the dynamics of LL fermions}\)

Sakita: 2 dim. context

Das, Dhar, Mandal, Wadia,...
Extend this to include fluctuating gauge fields by gauging $S_0$

$$\partial_t \rightarrow \hat{D}_t = \partial_t + i\hat{A}$$

$$S = \int dt \text{Tr} \left[ i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} - \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} - \hat{\rho}_0 \hat{U}^\dagger \hat{A} \hat{U} \right]$$

**gauge interactions**

In terms of bosonic fields

$$S = N \int dt \ d\mu \ \text{tr} \left[ i\rho_0 \ * \ U^\dagger \ * \ \partial_t U - \rho_0 \ * \ U^\dagger \ * \ (V + A) \ * \ U \right]$$

**QUESTION:** How is $A$ related to the gauge fields coupled to the original fermions?
Invariance under $U(N)$ rotations $\delta \hat{U} = -i \hat{\lambda} \hat{U}$ implies that $S$ is invariant under

$$\delta U = -i \lambda \ast U$$

(1)

$$\delta A(\vec{x}, t) = \partial_t \lambda(\vec{x}, t) - i (\lambda \ast (V + A) - (V + A) \ast \lambda)$$

Since $S$ describes gauge interactions it has to be invariant under usual gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda + i [ \vec{A}_\mu + A_\mu , \Lambda] , \quad \delta \vec{A}_\mu = 0$$

(2)

Background Perturbation

we should choose

$$A = \text{ function}(A_\mu, \vec{A}_\mu, V)$$

$$\lambda = \text{ function}(\Lambda, A_\mu, \vec{A}_\mu)$$

such that the gauge transformation (2) induces $\delta A$ in (1) (generalized Seiberg-Witten map)
Group theoretic analysis:

\[ \mathbb{CP}^k = \frac{SU(k + 1)}{U(k)} \quad (2k \text{ dim space}) \]

- \( U(k) \sim U(1) \times SU(k) \implies \) We can have both \( U(1) \) and \( SU(k) \) background magnetic fields

- Landau wavefunctions are functions on \( SU(k + 1) \) with particular transformation properties under \( U(k) \).

- Each Landau level forms an irreducible \( SU(k + 1) \) representation, whose degeneracy is easy to calculate.
Wavefunctions are functions on $SU(k+1)$; expressed in terms of Wigner $D$ functions
\[ \Psi \sim D_{L,R}^{(J)}(g) = \langle L \mid \hat{g} \mid R \rangle \]
quantum numbers of states in J rep.

Right/left transformations:
\[ \hat{R}_A \hat{g} = \hat{g} T_A, \quad \hat{L}_A \hat{g} = T_A \hat{g} \]

- $\hat{R}_a, \hat{R}_{k^2 + 2k} \rightarrow$ gauge transformations \((U(k))\)
- $\hat{R}_{+i}, \hat{R}_{-i} \rightarrow$ covariant derivatives \((i = 1, \cdots, k)\) \([\hat{R}_{+i}, \hat{R}_{-j}] \in U(k)\)
- $\hat{L}_A \rightarrow$ magnetic translations \((A \in SU(k+1))\)

How $\Psi$ transforms under $U(k)_R$ depends on choice of background fields

Choose "constant" $U(1)$ or $U(k)$ background magnetic fields.

\[ \bar{F} = d\bar{a} = n\Omega, \quad \Omega = \text{Kahler 2\text{\textendash\textquoteright} form} \]

(Field strengths $\sim U(k)$ structure constants in tangent frame basis $\sim$ Riemannian curvature)
\[ \Psi^J_{m;\alpha} \sim \langle m | \hat{g} | R \rangle \]

particular \( SU(k)_R \) repr. \( \tilde{J} \) with fixed \( U(1)_R \) charge \( \sim n \)

\[ m = 1, \cdots \dim \tilde{J} \implies \text{counts degeneracy of LL} \]

\[ \alpha = \text{internal gauge index} = 1, \cdots , N' = \dim \tilde{J} \]

- Hamiltonian

\[ H = \frac{1}{2Mr^2} \sum_{i=1}^{k} \hat{R}_{+i} \hat{R}_{-i} + \text{constant} \]

Lowest Landau level: \( \hat{R}_{-i} \Psi = 0 \)  
Holomorphicity condition

( \( | R \rangle \) is lowest weight state)
We know single particle spectrum $\rightarrow$ calculate symbols, star products

Calculate $S_0$ for large $N, K$ with $N \gg K \gg 1$ ($n \rightarrow \infty$ limit)

$$S_0 = N \int d\mu \, dt \left[ i(\rho_0 * U^\dagger * \partial_t U) - (\rho_0 * U^\dagger * V * U) \right]$$

A. Abelian background magnetic field $U(1)$

- Symbol of $\hat{X} \rightarrow$ function $X$
- $X * Y = XY + \frac{1}{n} \left( g^{ij} + \frac{i}{2} (\Omega^{-1})^{ij} \right) \partial_i X \partial_j Y + \cdots$
- $\left( [\hat{X}, \hat{Y}] \right)_{symbol} \rightarrow \frac{i}{n} \{X(\vec{x}, t), Y(\vec{x}, t)\}_{PB} = \frac{i}{n} (\Omega^{-1})^{ij} \partial_i X(\vec{x}, t) \partial_j Y(\vec{x}, t)$
- $\rho_0 \rightarrow \Theta \left( R_D^2 - r^2 \right)$, $R_D$ = droplet radius

April 2017 18 / 33
This is a \((2k - 1)\) (space) dimensional chiral action defined on the droplet boundary, with

\[
S_0 \sim \int_{\partial D} \left( \partial_t \phi + u \mathcal{L}\phi \right) \mathcal{L}\phi
\]

\[
\mathcal{L}\phi = (\Omega^{-1})^{ij}\hat{r}_j \partial_i \phi, \quad \mathcal{L} = \begin{cases} \text{derivative along boundary of droplet} \\ \rightarrow \partial_\theta \text{ in 2 dim.} \end{cases}
\]

B. Nonabelian background magnetic field \(U(k)\)

- Symbol of a matrix = \((N' \times N')\) matrix valued function

\[
\hat{X} \mapsto X_{\alpha\beta}(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_{m;\alpha}(\vec{x}) X_{ml}(t) \Psi^*_{l;\beta}(\vec{x})
\]

\(\alpha, \beta = 1, \cdots N'\)
Bosonic action can be written in terms of $G \in U(N')$

$$S_0 = \frac{1}{4\pi} \int_{\partial D} \text{tr} \left[ \left( G^\dagger \dot{G} + u G^\dagger \mathcal{L}G \right) G^\dagger \mathcal{L}G \right]$$

$$+ \frac{1}{4\pi} \int_D \text{tr} \left[ -d (i\tilde{A}dGG^\dagger + i\bar{A}G^\dagger dG) + \frac{1}{3} (G^\dagger dG)^3 \right] \left( \frac{\Omega}{2\pi} \right)^{k-1} \frac{1}{(k-1)!}$$

$$\equiv S_{WZW}(A^L = A^R = \bar{A})$$

$\mathcal{L} = (\Omega^{-1})^{ij} \hat{r}_j D_i = \text{covariant derivative along the boundary of droplet}$
In the presence of gauge fluctuations (DK)

\[
S = N \int dt \, d\mu \, \text{tr} \left[ i\rho_0 \ast U^\dagger \ast \partial_t U - \rho_0 \ast U^\dagger \ast (V + A) \ast U \right]
\]

\[= S_{\text{edge}} + S_{\text{bulk}}\]

\[S_{\text{edge}} \sim S_{\text{WZW}}(A^L = A + \tilde{A}, A^R = \tilde{A}) = \text{Chirally gauged WZW action in } 2k \text{ dim}\]

\[S_{\text{bulk}} \sim S_{\text{CS}}^{2k+1}(\tilde{A}) + \cdots = (2k + 1) \text{ dim CS action}\]

Here \(\tilde{A} = (A_0 + V, \bar{a}_i + \bar{A}_i + A_i) = \text{background + fluctuations}\)

Gauge Invariance \(\implies\) Anomaly Cancellation

\[\delta S_{\text{edge}} \neq 0, \quad \delta S_{\text{bulk}} \neq 0\]

\[\delta S_{\text{edge}} + \delta S_{\text{bulk}} = 0\]
What about metric perturbations?

- Universal matrix action describing dynamics within each Landau level
- single-particle spectrum + large $N$ limit $\implies$ (bulk + edge) effective actions
- gauge invariance is automatically built in

Questions:

- How important is precise knowledge of single particle spectrum?
- Can we deviate from $\mathbb{CP}^k$?
- How do we introduce metric perturbations?
The lowest Landau level obeys the holomorphicity condition

\[ \hat{R} - i \Psi = 0 \]

The number of normalizable solutions is given by the Dolbeault index.
The Dolbeault index is given by

\[
\text{Index}(\bar{\partial}_V) = \int_M \text{td}(T_{CM}) \wedge \text{ch}(V)
\]

\[ \text{td}(T_{CM}) = \text{Todd class (for complex tangent space)} = \text{given in terms of traces of curvatures} \]

\[ \text{ch}(V) = \text{Chern character} = \text{Tr} \left( e^{iF/2\pi} \right) \]
For a fully filled LLL (each particle carries $e = 1$):

degeneracy = charge $\implies$ index density = charge density

So we can use

$$\frac{\delta S_{\text{eff}}}{\delta A_0} = J_0 = \text{Dolbeault Index Density}$$

and integrate up to get $S_{\text{eff}}$. 


In 2+1 dimensions

\[ \text{Index}(\bar{\partial}_V) = \int_M \frac{iF}{2\pi} + \frac{iR}{4\pi} \]

Consider QHE on \( \mathbb{CP}^1 = SU(2)/U(1) \sim S^2 \)

The background values for the curvatures are

\[ \bar{F} = -i n\Omega, \quad \bar{R} \big|_{T_{MK}} = -i 2\Omega, \quad \Omega = i \left[ \frac{dzd\bar{z}}{1 + z\bar{z}} - \frac{\bar{z}dz \cdot d\bar{z}}{(1 + z\bar{z})^2} \right] \]

degeneracy of LLL= \( \text{Index}(\bar{\partial}_V) = n + 1 \) as expected

Charge density including fluctuations

\[ J_0 = \frac{iF}{2\pi} + \frac{iR}{4\pi} \]

This leads to the effective action

\[ S_{\text{LLL}}^{3d} = \frac{i^2}{4\pi} \int A \left[ F + R \right] + S_{\text{grav}} + \cdots \]
For higher Landau levels there is no holomorphicity condition, Dolbeault index is problematic.

The wave functions in the $s$-th LL for $\mathbb{C}P^1 = SU(2)/U(1) \sim S^2$ are

$$\Psi_m(g) \sim \langle J, m|g|J, -n/2 \rangle, \quad J = n/2 + s$$

They satisfy $R_3 \Psi = -n/2 \Psi$, but do not satisfy holomorphicity $R_- \Psi \neq 0$

The lowest weight state in the same representation

$$\tilde{\Psi}_m(g) \sim \langle J, m|g|J, -n/2 - s \rangle$$

which satisfy the holomorphicity condition $R_- \tilde{\Psi} = 0 \implies \text{LLL}$
It couples to a $U(1)$ background field

$$\mathcal{F} = -i(n + 2s)\Omega = \bar{F} + s\bar{R} = \bar{F} + \bar{R}_s$$

So we have a mapping

Particle with spin-0 in $s$-th LL $\implies$ Particle with spin-$s$ in LLL

counting of states same $\implies$ use Dolbeault index for degeneracy

Chern character: $\text{Tr} \left( e^{iF/2\pi} \right) \rightarrow \text{Tr} \left( e^{i(\mathcal{R}_s+F)/2\pi} \right)$
For $s$-th Landau level on $\mathbb{CP}^1 = SU(2)/U(1) \sim S^2$ we find
\[
\text{Index}(\bar{\partial}_V) = \int \left[ \frac{iF}{2\pi} + (s + \frac{1}{2}) \frac{iR}{2\pi} \right]
\]

$S_{\text{eff}}$ can be determined from this
\[
S_{3d}^{(s)} = \frac{i^2}{4\pi} \int A [F + (2s + 1)R] + S_{\text{grav}} + \cdots
\]

The purely gravitational term can be obtained from the gravitational part of the index density in $2k + 2$ dim following the descent approach
\[
[\text{Index Density}]_{2k+2} = d [\cdots]
\]

Derive the full topological effective action
\[
S_{3d}^{(s)} = \frac{i^2}{4\pi} \int \left\{ \left[ A + (s + \frac{1}{2}) \omega \right] d \left[ A + (s + \frac{1}{2}) \omega \right] - \frac{1}{12} \omega d\omega \right\}
\]

This agrees with Abanov, Gromov; Kletsov, Ma, Marinescu, Wiegmann; Bradlyn, Read; Can, Laskin, Wiegmann
Higher Dimensions

- In $2k + 1$ dimensions (fluctuations around $\mathbb{CP}^k = SU(k + 1)/U(k)$ background fields)
  - We have curvatures in the algebra of $U(k)$
    
    $$R = -i[R^0\mathbf{1} + \tilde{R}^a t_a], \quad R^0 = d\omega^0, \quad \tilde{R} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}$$
  - Abelian and/or nonabelian gauge fields
  - Nonzero spin to include higher Landau levels
    
    $$\mathcal{R}_s = -i[s R^0\mathbf{1} + \tilde{R}^a T_a]$$
  - The index theorem is now
    
    $$\text{Index}(\bar{\partial}_V) = \int_M td(T_c M) \wedge \text{Tr} \left( e^{i(\mathcal{R}_s + F)/2\pi} \right)$$
The full topological effective action is

$$S_{2k+1}^{(s)} = \int \left[ t d(T_cK) \wedge \sum_p (CS)_{2p+1}(\omega_s + A) \right]_{2k+1} + 2\pi \int \Omega_{2k+1}^{\text{grav}}$$

where

$$[td(T_cK) \wedge ch(S)]_{2k+2} = d\Omega_{2k+1}^{\text{grav}} + \frac{1}{2\pi} d \left[ td(T_cK) \wedge \sum_p (CS)_{2p+1}(\omega_s) \right]_{2k+1}$$

Gives correct expressions for degeneracies in all cases we know QHE wavefunctions: \(\mathbb{C}P^k\) (abelian and nonabelian gauge fields), \(S^2 \times S^2\), etc
2+1 dim, $s$-th Landau level

$$S^{(s)}_{3d} = \frac{i^2}{4\pi} \int \left\{ \left[ A + (s + \frac{1}{2}) \omega \right] d\left[ A + (s + \frac{1}{2}) \omega \right] - \frac{1}{12} \omega d\omega \right\}$$

4+1 dim, $s$-th Landau level

$\mathbb{CP}^2 = SU(3)/U(2); U(1)$ gauge field; $U(2)$ spin connection and curvature

$$S^{(s)}_{5d} = \frac{i^3(s + 1)}{(2\pi)^2} \int \left\{ \frac{1}{3!} \left( A + (s + 1)\omega^0 \right) \left[ d \left( A + (s + 1)\omega^0 \right) \right]^2 \\
- \frac{1}{12} \left( A + (s + 1)\omega^0 \right) \left[ (d\omega^0)^2 - \left[ ((s + 1)^2 - \frac{3}{2} \right] \text{Tr}(\tilde{R} \wedge \tilde{R}) \right] \right\}$$
\* 6+1 dim, lowest Landau level

\[ \mathbb{C}P^3 = SU(4)/U(3); \ U(1) \text{ gauge field; } U(3) \text{ curvature} \]

\[ S_{7d}^{\text{LLL}} = \frac{1}{(2\pi)^3} \int \left\{ \frac{1}{4!} \left( A + \frac{3}{2} \omega^0 \right) \left[ d \left( A + \frac{3}{2} \omega^0 \right) \right]^3 \right. \]

\[ - \frac{1}{16} \left( A + \frac{3}{2} \omega^0 \right) d \left( A + \frac{3}{2} \omega^0 \right) \left[ (d\omega^0)^2 + \frac{1}{3} \text{Tr}(\bar{R} \wedge \bar{R}) \right] \]

\[ + \frac{1}{1920} \omega^0 d\omega^0 \left[ 17(d\omega^0)^2 + 14\text{Tr}(\bar{R} \wedge \bar{R}) \right] + \frac{1}{720} \omega^0 \text{Tr}(\bar{R} \wedge \bar{R} \wedge \bar{R}) \right\} \]

\[ + \frac{1}{120} \int (CS)_{7}(\bar{\omega}) \]

- Extend QHE to higher dimensions; physical realizations of fuzzy spaces
- Universal matrix action → noncommutative field theory description of LL dynamics
- At large $N$ limit action describes dynamics of abelian/nonabelian quantum Hall droplets with gauge fluctuations
- anomaly free bulk/edge dynamics
- Use index theorems to introduce metric perturbations
- Applications to fluids and higher dim transport coefficients