Superconformal symmetry near horizons

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Material based on
J. B. Gutowski, GP, arXiv:1303.0869 ; U. Gran, J. B. Gutowski, GP, arXiv:1306.5765
S. W. Beck, J. B. Gutowski and GP, arXiv:1410.3431, arXiv:1601.06645
U. Gran, J. B. Gutowski, GP, arXiv:1607.00191
J. B. Gutowski and GP, arXiv:1702.06048

Conformal Symmetry

- Many of the recent developments in physics, like AdS/CFT, rely on the observation that
- (super)symmetry enhances near extreme BH and brane horizons and includes a conformal subgroup.

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Reissner-Nordström

Consider the Reissner-Nordström black hole [Carter]

$$ds^2 = -\frac{\Lambda}{\rho^2}dt^2 + \rho^2\Lambda^{-1}d\rho^2 + \rho^2 ds^2(S^2)$$

where

$$\Lambda = \rho^2 - 2M\rho + Q^2 = (\rho - \rho_+)(\rho - \rho_-)\,, \quad \rho_\pm = M \pm \sqrt{M^2 - Q^2}$$

Introduce Eddington-Finklestein coordinates as

$$d\rho^* = \rho^2\Lambda^{-1}d\rho\,, \quad u = t + \rho^*$$

to rewrite the metric as

$$ds^2 = -\frac{\Lambda}{\rho^2}du^2 + 2du\,d\rho + \rho^2 ds^2(S^2)$$

Next define the coordinate $r = \rho - \rho_+$ and observe that the metric is analytic in r. Expanding around $r = 0$

$$\begin{aligned} ds^2 &= 2du\left[dr - \frac{1}{2}\left(r\,\frac{\rho_+ - \rho_-}{\rho_+^2} + r^2\,\frac{2\rho_- - \rho_+}{\rho_+^3} + \mathcal{O}(r^3)\right)du\right] \\ &+ (\rho_+^2 + 2r\,\rho_+ + r^2)ds^2(S^2) \end{aligned}$$

- The linear term in r is the surface gravity of the horizon
- If the black hole is extreme $\rho_- = \rho_+$, then a near horizon limit can be defined by writing

$$u \to \ell^{-1}u\,, \quad r \to \ell r$$

and taking $\ell \to 0$

- The near horizon geometry in this case is

$$ds^2 = 2du\left[dr - \frac{1}{2}r^2 \frac{1}{\rho_+^2}du\right] + \rho_+^2 ds^2(S^2)$$

which is isometric to $AdS_2 \times S^2$.

- The symmetry enhances from $\mathbb{R} \times SO(3)$ to $SL(2,\mathbb{R}) \times SO(3)$
- Supersymmetry also enhances from 4 to 8 supersymmetries

Horizon symmetry enhancement

Let M be the near horizon geometry of an extreme, smooth, *Killing* horizon admitting a closed spatial horizon section $\mathcal{S}$ and preserving at least one supersymmetry, then

- Conjecture 1: The number of Killing spinors N of M are

$$N = 2N_- + \mathrm{Index}(D_E)$$

  where $N_- \in \mathbb{Z}_{>0}$, D_E is a Dirac operator twisted by E defined on the horizon sections $\mathcal{S}$. E depends on the gauge symmetries of supergravity.

- Conjecture 2: If M has non-trivial fluxes and $N_- \neq 0$, then M admits a $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra

Established for

- D=5 minimal gauged, D=11, IIB, heterotic, (massive) IIA supergravities and non-minimal gauged $\mathcal{N}$=2 D=4 supergravity

Remarks

- If the index vanishes, which is the case for non-chiral theories, then N is even. In particular for all odd dimensional horizons, N is even.
- The horizons with non-trivial fluxes of all non-chiral theories have a $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra
- If $N_- = 0$, then $N = \mathrm{index}(D_E)$ and so the number of Killing spinors is determined by the topology of horizons.
- The only symmetry assumptions are (i) a time-like Killing vector field which becomes null at the horizon, and (ii) that the horizon preserves one supersymmetry.

Consequences and Applications

These results can be applied in a variety of problems

- The existence of higher dimensional black holes with exotic topologies and geometries
  Asymptotically AdS_5 rings in minimal 5d gauged supergravity have been ruled out! [Grover, Gutowski, GP, Sabra; Grover, Gutowski, Sabra]
- Microscopic counting of entropy for black holes
  The presence of $\mathfrak{sl}(2,\mathbb{R})$ justifies the use of conformal mechanics in entropy counting.
- AdS/CFT: Provides a new method to classify all AdS backgrounds in supergravity.
- Geometry: A generalization of Lichnerowicz theorem for connections with GL holonomy.

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Horizon metric

Near a smooth extreme Killing horizon an (Eddington-Finklestein) coordinate system can be adapted such that the metric is [Isenberg, Moncrief; Friedrich, et al]

$$ds^2 = 2du[dr + r\, h_I(r,y)dy^I - \frac{1}{2}r^2 f(r,y)du] + \gamma_{IJ}(y,r)dy^I dy^J$$

In the near horizon limit, $u \to \ell^{-1}u$, $r \to \ell r$, $\ell \to 0$, the leading term of the metric is

$$ds^2 = 2du[dr + r\, h_I dy^I - \frac{1}{2}r^2\, \Delta du] + \gamma_{IJ}dy^I dy^J$$

where

$$h_I = h_I(0,y)\ , \quad \Delta = f|_{r=0}\ , \quad \gamma_{IJ} = \gamma_{IJ}(0,y)$$

- The near horizon metric has two isometries generated by translations in u and the scale transformation

$$u \to \ell^{-1} u \;, \quad r \to \ell r$$

- The two Killing vectors

$$\partial_u \;, \quad -u\partial_u + r\partial_r$$

  do not commute. The algebra of isometries is NOT $\mathfrak{sl}(2, \mathbb{R})$
- This coordinate system can be adapted in the presence of other fields like Maxwell and k-form gauge potentials
- The co-dimension 2 space given by $u = r = 0$ is the spatial horizon section, $\mathcal{S}$, and it is required to be compact without boundary.

M-horizons

The near horizon fields of D=11 supergravity are

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + d\tilde{s}^2(\mathcal{S}) ,$$
$$F = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + r\mathbf{e}^+ \wedge d_hY + X , \quad d_hY = dY - h \wedge Y ,$$

where

$$\mathbf{e}^+ = du , \quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du , \quad \mathbf{e}^i = e^i{}_J dy^J$$

The steps in the proof are as follows.

- Integration of KSEs along the lightcone directions r, u
- Independent KSEs on $\mathcal{S}$
- Horizon Dirac equations
- Two Lichnerowicz type of theorems
- Index and number of Killing spinors

Integrability of KSEs along the lightcone

The KSEs [Cremmer, Julia, Scherk] are

$$\mathcal{D}_M\epsilon = \nabla_M\epsilon - \left(\frac{1}{288}\Gamma_M{}^{L_1L_2L_3L_4}F_{L_1L_2L_3L_4} - \frac{1}{36}F_{ML_1L_2L_3}\Gamma^{L_1L_2L_3}\right)\epsilon = 0$$

These can be integrated along to lightcone directions to give

$$\epsilon = \epsilon_+ + \epsilon_- \ , \quad \Gamma_\pm\epsilon_\pm = 0 \ ,$$

with

$$\epsilon_+ = \eta_+, \qquad \epsilon_- = \eta_- + r\Gamma_-\Theta_+\eta_+ \ ,$$

and

$$\eta_+ = \phi_+ + u\Gamma_+\Theta_-\phi_-, \qquad \eta_- = \phi_- \ ,$$

where

$$\Theta_\pm = \left(\frac{1}{4}h_i\Gamma^i + \frac{1}{288}X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{12}Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2}\right) \ ,$$

and $\phi_\pm = \phi_\pm(y)$ do not depend on r or u.

Independent KSEs

The integration along the lightcone directions has two consequences. First after using the field equations and Bianchi identities, the remaining independent KSEs are

$$\nabla_i^{(\pm)}\phi_\pm \equiv \tilde{\nabla}_i\phi_\pm + \Psi_i^{(\pm)}\phi_\pm = 0 \ ,$$

where

$$\begin{aligned}\Psi_i^{(\pm)} &= \mp\frac{1}{4}h_i - \frac{1}{288}\Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4}X_{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{36}X_{i\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} \\ &\pm\frac{1}{24}\Gamma_i{}^{\ell_1\ell_2}Y_{\ell_1\ell_2} \mp \frac{1}{6}Y_{ij}\Gamma^j \ ,\end{aligned}$$

and $\tilde{\nabla}$ the Levi-Civita connection of $\mathcal{S}$.

Second, if ϕ_- is a solution, $\nabla_i^{(-)}\phi_- = 0$, then

$$\nabla_i^{(+)}\phi'_+ = 0 \ , \quad \phi'_+ = \Gamma_+\Theta_-\phi_-$$

Horizon Dirac operators

The associated horizon Dirac operators are

$$\mathcal{D}^{(\pm)}\phi_\pm \equiv \Gamma^i \nabla_i^{(\pm)}\phi_\pm = \Gamma^i \tilde{\nabla}_i \phi_\pm + \Psi^{(\pm)}\phi_\pm \ ,$$

where

$$\Psi^{(\pm)} = \Gamma^i \Psi_i^{(\pm)} = \mp\frac{1}{4} h_\ell \Gamma^\ell + \frac{1}{96} X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{8} Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2} \ .$$

Clearly,

$$\nabla_i^{(\pm)}\phi_\pm = 0 \Longrightarrow \mathcal{D}^{(\pm)}\phi_\pm = 0$$

The converse is also true, ie

$$\nabla_i^{(\pm)}\phi_\pm = 0 \Longleftrightarrow \mathcal{D}^{(\pm)}\phi_\pm = 0$$

A maximum principle

The proof of converse for the $\mathcal{D}^{(+)}$ operator relies on the formula that if $\mathcal{D}^{(+)}\phi_+ = 0$, then

$$\tilde{\nabla}^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 - h^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 = 2\langle \nabla^{(+)i}\phi_+, \nabla_i^{(+)}\phi_+\rangle \ .$$

Using the maximum principle for the function $\parallel \phi_+ \parallel^2$ based on the compactness of $\mathcal{S}$, one concludes that

$$\nabla_i^{(+)}\phi_+ = 0\ , \quad \parallel \phi_+ \parallel^2 = \text{const}\ .$$

which gives the proof of a Lichnerowicz type of theorem for $\mathcal{D}^{(+)}$

- A Lichnerowicz type of Theorem can also be demonstrated for the $\mathcal{D}^{(-)}$ operator

Index and supersymmetry

The spin bundle splits $S = S_+ \oplus S_-$ on $\mathcal{S}$ with respect to $\Gamma_\pm$, and $\mathcal{D}^{(+)} : \Gamma(S_+) \to \Gamma(S_+)$ and its adjoint $(\mathcal{D}^{(+)})^\dagger : \Gamma(S_+) \to \Gamma(S_+)$.
$\mathcal{D}^{(+)}$ has the same principal symbol as the Dirac operator and $\mathrm{Index}(\mathcal{D}^{(+)}) = 0$ as $\dim\mathcal{S} = 9$. Thus

$$\dim\ker\mathcal{D}^{(+)} = \dim\ker(\mathcal{D}^{(+)})^\dagger \; .$$

Then $(\mathcal{D}^{(+)})^\dagger\Gamma_+ = \Gamma_+\mathcal{D}^{(-)}$ and so

$$\dim\ker(\mathcal{D}^{(+)})^\dagger = \dim\ker\mathcal{D}^{(-)}$$

Thus

$$\dim\ker\mathcal{D}^{(+)} = \dim\ker\mathcal{D}^{(-)} \; .$$

The number of supersymmetries of a near horizon geometry is the number of parallel spinors of $\nabla^{(\pm)}$ and so from the Lichnerowicz theorems and the index

$$N = \dim\ker\mathcal{D}^{(+)} + \dim\ker\mathcal{D}^{(-)} = 2\dim\ker\mathcal{D}^{(-)} = 2N_- .$$

This proves that the number of supersymmetries preserved by M-horizon geometries is even.

Construction of ϕ_+ spinors from ϕ_- spinors

Recall that if $\nabla^{(-)}\phi_- = 0$, then

$$\nabla^{(+)}\phi_+ = 0\,, \quad \phi_+ = \Gamma_+\Theta_-\phi_-\,.$$

To find a second supersymmetry, $\phi_+ \neq 0$. Indeed after a partial integration argument and some use of the maximum principle

$$\mathrm{Ker}\,\Theta_- \neq \{0\} \Longleftrightarrow F = 0, h = \Delta = 0$$

So if $\mathrm{Ker}\,\Theta_- \neq \{0\}$, the near horizon geometries have vanishing fluxes and are products $\mathbb{R}^{1,1} \times S^1 \times X^8$, where X^8 has holonomy contained in $Spin(7)$.

- For horizons with non-trivial fluxes if $\phi_- \neq 0$, then $\phi_+ = \Gamma_+\Theta_-\phi_- \neq 0$

$\mathfrak{sl}(2,\mathbb{R})$ symmetry

Every near horizon geometry with non-trivial fluxes admits at least two Killing spinors given by

$$\epsilon_1 = \epsilon(\phi_-, 0)\ , \quad \epsilon_2 = \epsilon(\phi_-, \phi_+)\ , \quad \phi_+ = \Gamma_+ \Theta_- \phi_-$$

These give rise to 3 Killing vector bi-linears given by

$$\begin{aligned} K_1 &= -2u \parallel \phi_+ \parallel^2 \partial_u + 2r \parallel \phi_+ \parallel^2 \partial_r + V^i \tilde{\partial}_i\ , \\ K_2 &= -2 \parallel \phi_+ \parallel^2 \partial_u\ , \\ K_3 &= -2u^2 \parallel \phi_+ \parallel^2 \partial_u + (2 \parallel \phi_- \parallel^2 + 4ru \parallel \phi_+ \parallel^2)\partial_r + 2uV^i \tilde{\partial}_i\ , \end{aligned}$$

where V is a Killing vector on $\mathcal{S}$ which leaves all the data invariant.
They satisfy the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$$[K_1, K_2] = 2 \parallel \phi_+ \parallel^2 K_2\ , \quad [K_2, K_3] = -4 \parallel \phi_+ \parallel^2 K_1\ , \quad [K_3, K_1] = 2 \parallel \phi_+ \parallel^2 K_3\ .$$

- If $V_i = \langle \Gamma_+ \phi_-, \Gamma_i \phi_+ \rangle = 0$, the near horizon geometries of M-theory are $AdS_2 \times_w \mathcal{S}$

Supersymmetry of AdS backgrounds

- The classification of AdS supergravity backgrounds [Freund-Rubin] is a longstanding problem
- Includes applications in AdS/CFT, and in string, M-theory, and supergravity compactifications
- Warped, flux compactification to Minkowski space also arise in the limit of infinite AdS radius.
- The aim is to classify all AdS backgrouds

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Assumptions

- No assumptions are made on the form of the fields or that of the Killing spinors apart from imposing the symmetries of AdS spaces on the former
- AdS backgrounds are special cases of near horizon geometries
- The counting uncovers new Lichnerowicz type of theorems
- The novelty of the approach is that it is completely general

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AdS backgrounds

The a priori number of sypersymmetries preserved by D=11, IIB and IIA AdS backgrounds are [Beck, Gutowski, GP]

AdS_n	N
$n=2$	$2k, k<16$
$n=3$	$2k, k<16$
$n=4$	$4k, k\leq 7, 32(D=11)$
$n=5$	$8, 16, 24, 32(IIB)$
$n=6$	16
$n=7$	$16, 32(D=11)$

Table: The proof for AdS_2 requires the maximum principle. For the rest, no such assumption is necessary. The bounds on k arise from the classification of solutions with near maximal [Gran, Gutowski, GP] and maximal [Figueroa O' Farrill, GP]supersymmetry.

Sketching the proof

The warp, flux, AdS backgrounds are special cases of near horizon geometries

$$ds^2 \;=\; 2du(dr + rh) + A^2\Big(dz^2 + e^{2z/\ell}\sum_{a=1}^{n-3}(dx^a)^2\Big) + ds^2(M^{11-n}) \; ,$$

with

$$h = -\frac{2}{\ell}dz - 2A^{-1}dA, \quad \Delta = 0 \; ,$$

A is the warp factor and ℓ the AdS radius.

- Solve the KSEs along the lightcone directions as for horizons
- solve the KSEs along z and then the remaining x^a coordinates
- count the multiplicity of Killing spinors

The solution along the lightcone directions can be done as in the horizon case, while the solution along the z and x^a coordinates gives

$$\phi_+ = \sigma_+ - \frac{1}{\ell} x^a \Gamma_{\rm a} \Gamma_z \tau_+ + e^{-z/\ell} \tau_+ \ , \quad \phi_- = \sigma_- + e^{z/\ell} (-\frac{1}{\ell} x^a \Gamma_{\rm a} \Gamma_z \sigma_- + \tau_-) \ ,$$

The remaining independent KSEs on M^{11-n} are

$$D_i^{(\pm)} \sigma_\pm = 0 \ , \quad D_i^{(\pm)} \tau_\pm = 0 \ ,$$

and

$$\mathcal{A}^{(\pm)} \sigma_\pm = 0 \ , \quad \mathcal{B}^{(\pm)} \tau_\pm = 0 \ ,$$

- The integration over z introduces new algebraic KSEs

The counting

To count the multiplicity, it turns out that if $\sigma_\pm$ is a solution, so is

$$\tau_\pm = \Gamma_{z\mathrm{a}}\sigma_\pm$$

and vice-versa

Similarly, if σ_+, τ_+ is a solution, so is

$$\sigma_- = A\Gamma_-\Gamma_z\sigma_+ \ , \quad \tau_- = A\Gamma_-\Gamma_z\tau_+$$

and vice-versa.

Furthermore, if σ_- is Killing spinor, then

$$\sigma'_+ = \Gamma_{\mathrm{ab}}\sigma_+ \ , \quad \mathrm{a} < \mathrm{b} \ ,$$

is also a Killing spinor.

- The number of supersymmetries are derived by counting the linearly independent solutions

New Lichnerowicz theorems

One can establish new Lichnerowicz type theorems as

$$\mathscr{D}^{(\pm)}\sigma_\pm = 0 \Longleftrightarrow D_i^{(\pm)}\sigma_\pm = 0\,, \quad \mathcal{A}^{(\pm)}\sigma_\pm = 0\,,$$

These are based on maximum principle formulae

$$D^2 \parallel A^{-1}\sigma_- \parallel^2 + nA^{-1}\partial^i A\partial_i \parallel A^{-1}\sigma_- \parallel^2 = 2A^{-2}\langle \mathbb{D}_i^{(-)}\sigma_-, \mathbb{D}^{(-)i}\sigma_-\rangle$$
$$+2\frac{9n-18}{11-n}A^{-2} \parallel \mathcal{A}^{(-)}\psi \parallel^2\,,$$

where $\mathbb{D}_i^{(-)} = D_i^{(-)} + \frac{2-n}{11-n}\Gamma_i\mathcal{A}^{(-)}$.

- If the solution is smooth, the warp factor A is nowhere zero.

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Classification of Warped AdS

One of the issues that arise in the classification of warped AdS backgrounds [Gran, Gutowski, GP] is that the metric on AdS_{k+1} can be written as a warped product $AdS_k \times_w \mathbb{R}$ [strominger]

$$ds^2(AdS_{k+1}) = \ell^2 dy^2 + \ell^2 \cosh^2 y\, ds^2(AdS_k)\ , \quad y \in \mathbb{R}\ ,$$

- Any $AdS_n \times_w M^{D-n}$ solution can be re-interpreted as a $AdS_k \times_w M^{D-k}$ solution for $k < n$.
- The Killing spinors of AdS backgrounds do not factorize into Killing spinors on AdS and Killing spinors on the internal space. This is particularly obvious for $\mathbb{R}^k \times_w M^{D-k}$ solutions.
- D=11 supergravity admits $AdS_k \times_w M^{11-k}$ maximally supersymmetric solutions for $k \leq 7$. Similar results apply to other theories.
- There are de Sitter supersymmetric solutions

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Classification of Warped AdS

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$$ds^2(AdS_{k+1}) = \ell^2 dy^2 + \ell^2 \cosh^2 y \, ds^2(AdS_k) \ , \quad y \in \mathbb{R} \ ,$$

- Any $AdS_n \times_w M^{D-n}$ solution can be re-interpreted as a $AdS_k \times_w M^{D-k}$ solution for $k < n$.
- The Killing spinors of AdS backgrounds do not factorize into Killing spinors on AdS and Killing spinors on the internal space. This is particularly obvious for $\mathbb{R}^k \times_w M^{D-k}$ solutions.
- D=11 supergravity admits $AdS_k \times_w M^{11-k}$ maximally supersymmetric solutions for $k \leq 7$. Similar results apply to other theories.
- There are de Sitter supersymmetric solutions

AdS_5 , $N = 24$

Theorem: There are no smooth AdS_5 solutions preserving $N = 24$ supersymmetries with compact without boundary internal space in all type II and $D = 11$ supergravities.

There are plenty of AdS_5 solutions apart from the IIB $AdS_5 \times S^5$ preserving less supersymmetry [Gauntlett, Martelli, Sparks, Waldram; Piltch, Warner; Maldacena Nunez; Itsios, Nunez, Sfetsos, Thomson] and a more systematic investigation was done by [Apruzzi, Fazzi, Passias, Tomasiello].

Proof: Consider the D=11 case, $AdS_5 \times_w M^6$. Use maximum principle to establish that $\| \sigma_+ \|$ is constant. From the algebraic KSE

$$i_W \star_6 X = 6 \| \sigma_+ \|^2 dA ,$$

X 4-form field strength, W isometries, and A warped factor. This implies

$$i_W dA = 0$$

Then the homogeneity theorem [Figueroa O'Farrill, Hustler] gives A constant and $X = 0$.

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AdS_6

Theorem: There are no smooth AdS_6 solutions with compact without boundary internal space in (massive) type IIA and $D = 11$ supergravities.

However there are AdS_6 solutions in massive IIA [Brandhuber, Oz] and IIB D'Hoker, Gutperle, Uhlemann]; a systematic investigation [Apruzzi, Fassi, Passias, Rosa, Tomassiello]

Proof: Set $\Lambda = \sigma_+ + \tau_+$, gravitino yields

$$D_i \parallel \Lambda \parallel^2 = -A^{-1} D_i A \parallel \Lambda \parallel^2 + \frac{1}{6} \star_5 X_i \langle \Lambda, \Gamma_{(4)} \Lambda \rangle \ , \quad \Gamma_{(4)} = \Gamma_{123z}$$

Algebraic and gravitino KSEs give

$$D^i D_i \parallel \Lambda \parallel^2 + 2A^{-1} D^i A D_i \parallel \Lambda \parallel^2 = 0 \ .$$

In turn maximum principle implies that $\parallel \Lambda \parallel$ is constant.
The algebraic KSE gives $\langle \sigma_+, \Gamma_{(4)} \sigma_+ \rangle = 0$, so A is constant, and

$$D_i \langle \sigma_+, \Gamma_{(4)} \sigma_+ \rangle = \frac{1}{6} \star_5 X_i \parallel \sigma_+ \parallel^2 \ ,$$

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Heterotic

Theorem: In heterotic theory with $dH = 0$

- There are no AdS_n, $n > 3$, supersymmetric backgrounds
- There are no smooth AdS_2 backgrounds for which the internal space is compact without boundary
- AdS_3 backgrounds preserve 2,4,6 and 8 supersymmetries
- Smooth AdS_3 backgrounds preserving 8 supersymmetries with compact without boundary internal space are locally isometric to either $AdS_3 \times S^3 \times T^4$ or $AdS_3 \times S^3 \times K_3$
- Although there is no classification of all possible backgrounds, there is a clear overview of all possibilities and what equations should be solved to achieve the task.

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Geometry

The geometry of AdS_3 backgrounds is as follows:

N	M^7	B^k	*fibre*
2	G_2	–	–
4	$SU(3)$	$U(3)$	S^1
6	$SU(2)$	$self-dual-Weyl$	S^3
8	$SU(2)$	$hyper-Kahler$	S^3

Table: The G-structure of M^7 is compatible with a connection with skew-symmetric torsion. For $N = 4, 6, 8$, M^7 is a local (twisted) fibration over a base space B^k with fibre either S^1 or S^3 . The base spaces B are conformally balanced with respect to the associated fundamental forms.

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Conclusion

- The horizon conjecture states that the emergence of conformal symmetry near supersymmetric Killing horizons is a consequence of regularity. It does not depend of the choice of supergravity theory.
- There is much progress in the classification of AdS backgrounds. Though the task has not been completed as yet.
- Global techniques, index theorem and maximum principle, give new insights into the properties of supersymmetric backgrounds.
- There are applications to geometry like the new Lichnerowicz type of theorems
- There are several non-existence theorems for smooth AdS backgrounds with compact internal space. Though their applications in AdS/CFT have not fully explored.

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