Frame Covariance in Scalar-Curvature Theories of Inflation

Sotirios Karamitsos
[sotirios.karamitsos@manchester.ac.uk]

(based on work in progress with Apostolos Pilaftsis)

Consortium for Fundamental Physics, School of Physics and Astronomy,
University of Manchester

HEP 2017
April 8, 2017
Ioannina, Greece
Introduction

Classical frame covariance

Quantum perturbations in field space

Specific models

Frame invariant radiative corrections
Introduction

- Inflation originally proposed to resolve *flatness* and *horizon* problems; also an excellent generic explanation for the origin of cosmological anisotropies.

- Era of precision cosmology
  - Increasingly clear that minimal models are not viable
  - Radiative corrections can and will affect observables
  - Frame problem beyond the Born approximation has palpable effect on phenomenology

- Desirable to develop a manifestly *frame-covariant* formalism to generate invariant predictions beyond the tree level
Action $S = S[g_{\mu\nu}, \varphi, f(\varphi), k(\varphi), V(\varphi)]$ for wide class of inflation models given in *Jordan frame* by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{f(\varphi)}{2}R + \frac{1}{2}k_{AB}(\varphi)g^{\mu\nu}(\nabla_\mu \varphi^A)(\nabla_\nu \varphi^B) - V(\varphi) \right]$$

Frame transformations are compositions of

- **Conformal transformations**
  $$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \varphi^A \rightarrow \tilde{\varphi}^A = \Omega^{-1} \varphi^A$$

- **Reparametrisations** of the inflaton
  $$\varphi^A \rightarrow \tilde{\varphi}^A = \varphi^A(\varphi^B), \quad \left( \frac{d\tilde{\varphi}^A}{d\varphi^B} \right) = \Omega^{-1} K^A_B(\varphi)$$

**Frame problem**: are frame transformations physically meaningful?

- Consensus for frame invariance at tree level; situation beyond the tree level is more nuanced
- Radiative corrections can be calculated in terms of effective model functions; hence, a frame-covariant theory at the classical level is desirable
Under an active frame transformation, model functions transform as
\[
\tilde{f}(\varphi) = \Omega^{-2} f(\varphi),
\]
\[
\tilde{k}_{AB} = [k_{AB} - 6f(\ln \Omega),_A (\ln \Omega),_B + 3f,_,A (\ln \Omega),_B + 3(\ln \Omega),_Af,_,B] K_A^A K_B^B,
\]
\[
\tilde{V}(\varphi) = \Omega^{-4} V(\varphi)
\]
Form of action does not change (models related by frame transformation define an equivalence class): starting point of frame covariance
\[
S[g_{\mu\nu}, \varphi, f(\varphi), k(\varphi), V(\varphi)] = S[\tilde{g}_{\mu\nu}, \tilde{\varphi}, \tilde{f}(\tilde{\varphi}), \tilde{k}(\tilde{\varphi}), \tilde{V}(\tilde{\varphi})]
\]
A quantity is characterized as frame covariant if it transforms as
\[
\tilde{X}_{\tilde{A}_1 \tilde{A}_2 \ldots \tilde{A}_p}^{\tilde{B}_1 \tilde{B}_2 \ldots \tilde{B}_q} = \Omega^{-(m_x+p-q)} (K_{A_1}^{A_1} K_{A_2}^{A_2} \ldots) X_{A_1 A_2 \ldots A_p}^{A_1 A_2 \ldots A_p} (K_{B_1}^{B_1} K_{B_2}^{B_2} \ldots)
\]
Equations of motion not *manifestly* frame covariant

Define **conformally** covariant and **reparametrization** covariant derivative

\[
X^{A_1 A_2 \ldots A_p}_{B_1 B_2 \ldots B_q, C} \equiv \ X^{A_1 A_2 \ldots}_{B_1 B_2 \ldots, C} - \frac{m_X f}{2} X^{A_1 A_2 \ldots}_{B_1 B_2 \ldots, C} + \frac{\Gamma^{A_1}_{CD} X^{D A_2 \ldots A_p}_{B_1 B_2 \ldots B_q}}{f} + \frac{\Gamma^{A_2}_{CD} X^{A_1 D \ldots A_p}_{B_1 B_2 \ldots B_q}}{f} + \cdots \\
- \frac{\Gamma^{D}_{B_1 C} X^{A_1 A_2 \ldots A_p}}{f} - \frac{\Gamma^{D}_{B_1 C} X^{A_1 A_2 \ldots A_p}}{f} - \cdots
\]

Introduce covariant quantities with \( \mathcal{D}_\lambda X^{A_1 \ldots}_{B_1 \ldots} \equiv X^{A_1 \ldots}_{B_1 \ldots, C} \ (d\varphi^C / d\lambda) \)

\[
G_{AB} \equiv \frac{k_{AB}}{f} + \frac{3 f_A f_B}{2 f^2} , \quad U \equiv \frac{V}{f^2} , \quad \mathcal{H} = \mathcal{D}_t \ln a,
\]

Cosmological equations of motion written in frame covariant manner

\[
\mathcal{D}_t \mathcal{D}_t \varphi^A + 3 \mathcal{H} (\mathcal{D}_t \varphi^A) + f G^{AB} U_{,B} = 0 , \quad \mathcal{H}^2 = \frac{1}{3} \left( \frac{G_{AB} (\mathcal{D}_t \varphi^A) (\mathcal{D}_t \varphi^B)}{2} + f U \right)
\]
Quantum perturbations

- Anisotropies seeded by correlation functions of inflaton perturbations $\delta \varphi^A$ and metric perturbations $g_{\mu\nu}dx^\mu dx^\nu = (1 + 2\Phi)N_L^2 dt^2 + \ldots$

$$Q^A \equiv \delta \varphi^A + \frac{D_t \varphi^A}{\mathcal{H}} \Phi$$

- **Field space**: manifold where inflationary trajectories live
  - $\varphi^A$ take on the role of coordinates; $G_{AB}$ takes on the role of a metric
  $$d\sigma^2 = G_{AB} d\varphi^A d\varphi^B$$

- With vielbein formalism, we decompose perturbations into *adiabatic* (parallel) and *entropic* (perpendicular) components ($s^A_B \equiv \delta^A_B - e^A_{\sigma} e_{\sigma B}$)

$$e_{\sigma}^A = \frac{D_t \varphi^A}{D_t \sigma}, \quad e_{s}^A = \frac{s^A_B U^B;B}{\sqrt{s_{AB} U^A;A U^B}}$$

- Comoving adiabatic/entropic curvature perturbation given by

$$\mathcal{R} \equiv \frac{\mathcal{H}}{D_t \sigma} Q^\sigma, \quad S \equiv \frac{\mathcal{H}}{D_t \sigma} Q^s$$
Quantum perturbations

- With $\delta N$ formalism, two-point function of $\mathcal{R}$ given by
  \[ P_{\mathcal{R}} \equiv N_{A}N_{B} \langle Q^{A} | Q^{B} \rangle \]

- Equation of motion for perturbations is
  \[ \mathcal{D}_{t} \mathcal{D}_{t} Q^{A}_{k} + 3\mathcal{H} \mathcal{D}_{t} Q^{A}_{k} + \frac{k^{2}}{a^{2}} Q^{A}_{k} + M^{A}_{B} Q^{B}_{k} = 0, \]

  \[ M_{AB} \equiv fU;_{AB} - R_{ABCD}(\mathcal{D}_{t} \varphi^{C})(\mathcal{D}_{t} \varphi^{D}) - \frac{1}{N_{La}^{3}} \mathcal{D}_{t} \left[ \frac{N_{La}^{3}}{\mathcal{H}} (\mathcal{D}_{t} \varphi_{A})(\mathcal{D}_{t} \varphi_{B}) \right] \]

- Define frame invariant Hubble slow roll hierarchy
  \[ \bar{\epsilon}_{H,n} \equiv -\mathcal{D}_{N} \ln \epsilon_{H,n-1}, \]
  \[ \bar{\epsilon}_{H,1} \equiv -\mathcal{D}_{N} \ln \mathcal{H} \]

- Canonically quantizing $Q^{A}$ and imposing Bunch-Davies vacuum condition for very early times returns the dimensionless power spectra ($\bar{\epsilon}_{H,1} \equiv \bar{\epsilon}_{H}$)
  \[ P_{\mathcal{R}} \equiv \frac{1}{8\pi^{2}} \frac{\mathcal{H}^{2}}{f(\varphi)\bar{\epsilon}_{H}}, \quad P_{T} \equiv \frac{2}{\pi^{2}} \frac{\mathcal{H}^{2}}{f(\varphi)} \]

Sotirios Karamitsos
University of Manchester
Frame Covariance in Scalar-Curvature Theories of Inflation
Quantum perturbations

- Beyond horizon exit, $S$ continues to evolve

$$\mathcal{D}_t \mathcal{R} = A \mathcal{H} S,$$

$$\mathcal{D}_t S = B \mathcal{H} S$$

- First entropic mode $S$ (parallel to field-space acceleration) couples to $\mathcal{R}$

$$\begin{pmatrix} \mathcal{R} \\ S \end{pmatrix} = \begin{pmatrix} 1 & T_{RS} \\ 0 & T_{SS} \end{pmatrix} \begin{pmatrix} \mathcal{R}_* \\ S_* \end{pmatrix},$$

$$T_{SS} = \exp \left( \int_{t_*}^{t} dt' B \mathcal{H} \right), \quad T_{RS} = \exp \left( \int_{t_*}^{t} dt' T_{SS} A \mathcal{H} \right)$$

- Details of entropy transfer depend on post-inflation processes (such as reheating), but its effects can be encoded into transfer angle $\Theta$, suppressed when turn rate $\omega \equiv |\mathcal{D}_t \mathcal{D}_\sigma \varphi^A|$ in field space is small

$$P_{\mathcal{R}}(t) = P_{\mathcal{R}}(t_*) \cos^{-2} \Theta$$
Cosmological observables may be calculated under the generalized slow roll hierarchy $\mathcal{D}_t \mathcal{D}_t \varphi^A \ll \mathcal{H}(\mathcal{D}_t \varphi^A)$:

$$
n_R - 1 \equiv \left. \frac{d \ln P_R}{d \ln k} \right|_{k = a \mathcal{H}}, \quad n_T \equiv \left. \frac{d \ln P_T}{d \ln k} \right|_{k = a \mathcal{H}}, \quad r \equiv \frac{P_T}{P_R},$$

$$
\alpha_R \equiv \left. \frac{d n_R}{d \ln k} \right|_{k = a \mathcal{H}}, \quad \alpha_T \equiv \left. \frac{d n_T}{d \ln k} \right|_{k = a \mathcal{H}}, \quad f_{NL} \equiv \frac{5}{6} \frac{N;^A N;^B N;^{AB}}{(N;^A N;^A)^2}.$$

Predictions for observables may be written in terms of Hubble slow roll parameters for slowly varying $\Theta$ ($\bar{\epsilon}_H, \bar{\eta}_H, \bar{\xi}_H$) as

$$
n_R = 1 - 2\bar{\epsilon}_H - \bar{\eta}_H, \quad n_T = -2\bar{\epsilon}_H, \quad r = 16\bar{\epsilon}_H \cos^2 \Theta,$$

$$
\alpha_R = -2\bar{\epsilon}_H \bar{\eta}_H - \bar{\eta}_H \bar{\xi}_H, \quad \alpha_T = -2\bar{\epsilon}_H \bar{\eta}_H, \quad f_{NL} \equiv \frac{5}{6} \frac{N;^A N;^B N;^{AB}}{4\bar{\epsilon}_H^2}.$$

Consistency relation for running of indices:

$$
(1 + n_T - n_R) r = -8\alpha_T \cos^2 \Theta$$
Quantum perturbations

- Slow-roll hierarchy for inflaton fields constrains us to the inflationary attractor class of solutions:

\[ \mathcal{H}^2 = \frac{fU}{3}, \quad \mathcal{D}_N \varphi^A = G^{AB} (\ln U)_{;B} \]

- This leads to potential slow roll parameter hierarchy; can be used to write observables in terms of \( \varphi^A \) via by \( \bar{\epsilon}_{H,n} \rightarrow \bar{\epsilon}_{U,n} \)

\[ \bar{\epsilon}_{U,n} \equiv - (\ln \bar{\epsilon}_{U,n-1})_A G^{AB} (\ln U)_{;B}, \]

\[ \bar{\epsilon}_{U,1} \equiv \frac{1}{2} G^{AB} (\ln U)_{;A} (\ln U)_{;B} \]

- Slow-roll implies large isocurvature effective mass \( m_s^2 \gg \mathcal{H}^2 \); isocurvature perturbations dampened on the geodesic

- End-of-inflation condition \( \epsilon_H = 1 \) defines \( (n - 1) \)-dimensional surface in field space; must select an inflationary trajectory before predictions can be made for any given model
Specific models

- Higgs inflation-inspired two-field model with $f = M_P^2 + \xi \phi^2$ and $V = (\lambda/4)\phi^4 + (m^2/2)\chi^2$
- Range of $\xi$ limited by normalization of power spectrum:
  - for nominal values $m^2/(\lambda M_P^2) = 1$, $\lambda = 10^{-12}$, $\xi_{\text{max}} \approx 0.0113$
  - for $\xi = 0.01$, $\phi_0 \approx 0.681 M_P$

![Graph showing predicted and observed $P_R/M_P^4$ vs $\phi_0/M_P$]

<table>
<thead>
<tr>
<th></th>
<th>predicted</th>
<th>PLANCK 2015</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.055</td>
<td>$\leq 0.08$ (95% CL)</td>
</tr>
<tr>
<td>$n_{\mathcal{R}}$</td>
<td>0.960442</td>
<td>$0.965 \pm 0.006$ (68% CL)</td>
</tr>
<tr>
<td>$\alpha_{\mathcal{R}}$</td>
<td>$-0.000786$</td>
<td>$-0.008 \pm 0.008$ (68% CL)</td>
</tr>
<tr>
<td>$\alpha_T$</td>
<td>$-0.000225$</td>
<td>$-0.000155 \pm 0.00016^*$ (68% CL)</td>
</tr>
<tr>
<td>$f_{NL}$</td>
<td>$-0.000972$</td>
<td>$0.8 \pm 5.0$ (68% CL)</td>
</tr>
</tbody>
</table>

- May study $F(\phi, R)$ theories as scalar-curvature theories via Lagrange transformation
Higher loop contributions to observables usually argued to be highly suppressed

- Assume metric corrections are suppressed and inflaton does not couple to matter fields
- Apply Vilkovisky–De Witt formalism including conformal transformations to calculate radiative corrections

Promote frame covariance to configuration space: \( A \rightarrow a \equiv (A, x_A) \)

\[
\tilde{X}^{\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_p}_{\tilde{b}_1 \tilde{b}_2 \ldots \tilde{b}_q} = \Omega^{-(m_x + p - q)} (K^{\tilde{a}_1}_{\tilde{a}_1} K^{\tilde{a}_2}_{\tilde{a}_2} \ldots) X^{a_1 a_2 \ldots a_p}_{b_1 b_2 \ldots b_q} (K^{b_1}_{\tilde{b}_1} K^{b_2}_{\tilde{b}_2} \ldots)
\]

Frame-covariant effective action ordinarily given by

\[
\exp \left( \frac{i}{\hbar} \Gamma[\Phi] \right) \equiv \int [\mathcal{D}\varphi^A] \mathcal{M}[\varphi] \exp \left\{ \frac{i}{\hbar} \left[ S[\varphi] - \frac{\delta \Gamma}{\delta \varphi^a} (\Phi^a - \varphi^a) \right] \right\}
\]

Standard formula privileges particular parametrization of fields; replacing \((\Phi^a - \varphi^a)\) with covariant two-point function \(\sigma^a(\Phi^b - \varphi^b)\) is equivalent to redefining mean field as belonging to tangent space
Frame invariant radiative corrections

- Expand effective action equation to find one-loop effective action:

\[ \Gamma_1[\varphi] = \ln \mathcal{M}[\varphi] - \frac{1}{2} \ln \det \left( \frac{\delta^2 S[\varphi]}{\delta \varphi^a \delta \varphi^b} \right) \]

- Replace \( \delta / \delta \varphi^a \) by covariant functional derivative; achievable simply by promoting field space indices to configuration space indices

\[ \nabla_c X_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p} \equiv \frac{\delta}{\delta \varphi^c} X_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p} - \frac{m_X}{2} \frac{f_{,c}}{f} X_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p} + \Gamma_{cd} X_{b_1 b_2 \ldots b_q}^{a_1 a_2 \ldots a_p} + \cdots \]

\[ -\Gamma_{b_1 c} X_{d b_2 \ldots b_q}^{a_1 a_2 \ldots a_p} - \Gamma_{b_1 d} X_{b_1 d \ldots b_q}^{a_1 a_2 \ldots a_p} - \cdots \]

- Configuration space metric and measure chosen appropriately

\[ G_{(A,x)(B,y)} \equiv G_{AB} \delta(x - y), \quad \mathcal{M}[\varphi] \equiv \sqrt{\det G_{(A,x)(B,y)}} \]

- With above specifications, \( \Gamma_1[\varphi] \) is invariant, leading to manifestly invariant first order radiative corrections
Conclusion

- Fully frame covariant formalism is introduced, incorporating conformal transformations; applies to all scalar-curvature theories as well as $F(\varphi, R)$ theories.

- Well-known techniques from differential geometry employed to distinguish between adiabatic/entropy perturbations and study their phenomenology in a covariant way.

- Nonminimal Higgs inflation-inspired model used to illustrate; straightforward to calculate observables, up to transfer between adiabatic and entropic modes.

- Radiative corrections take into account by direct extension to configuration space: effective action is fully frame invariant, leading to frame-invariant radiative corrections.

- Possible extension to include gravitational radiative corrections.